CK-12

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CHAPTER 3

CHAPTER 3 Parallel and Perpendicular Lines

Chapter Outline

In this chapter, you will explore the different relationships formed by parallel and perpendicular lines and planes. Different angle relationships will also be explored and what happens to these angles when lines are parallel. You will continue to use proofs, to prove that lines are parallel or perpendicular. There will also be a review of equations of lines and slopes and how we show algebraically that lines are parallel and perpendicular.

3.1 Lines and Angles

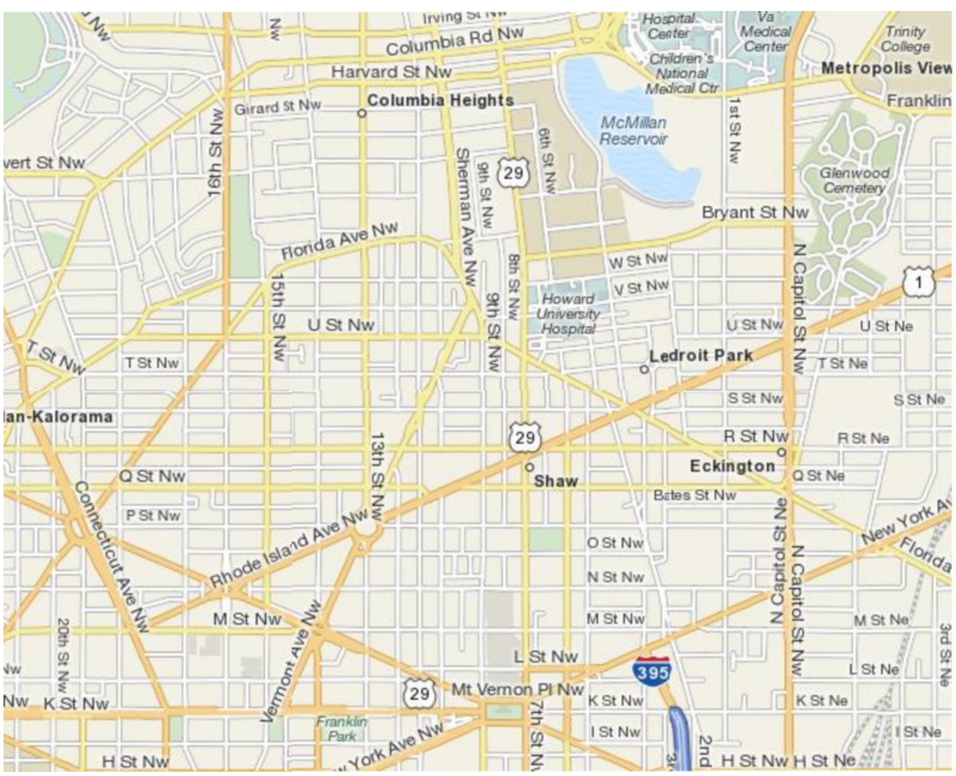
Learning Objectives

- Identify parallel lines, skew lines, and parallel planes.
- Use the Parallel Line Postulate and the Perpendicular Line Postulate.
- Identify angles made by transversals.

Review Queue

a. What is the equation of a line with slope -2 and passes through the point (0, 3)?
b. What is the equation of the line that passes through (3, 2) and (5, -6).
c. Change $4x - 3y = 12$ into slope-intercept form.
d. Are $y = \frac{1}{3}x$ and $y = -3x$ perpendicular? How do you know?

Know What? To the right is a partial map of Washington DC. The streets are designed on a grid system, where lettered streets, A through Z run east to west and numbered streets 1^{st} to 30^{th} run north to south. Just to mix things up a little, every state has its own street that runs diagonally through the city. There are, of course other street names, but we will focus on these three groups for this chapter. Can you explain which streets are parallel and perpendicular? Are any skew? How do you know these streets are parallel or perpendicular?

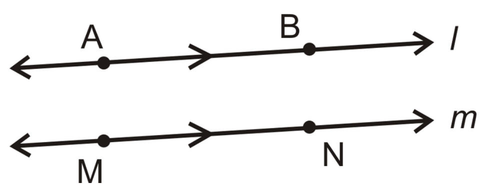


If you are having trouble viewing this map, check out the interactive map here: http://www.travelguide.tv/washington/map.html

Defining Parallel and Skew

Parallel: When two or more lines lie in the same plane and never intersect.

The symbol for parallel is $||$. To mark lines parallel, draw arrows ($>$) on each parallel line. If there are more than one pair of parallel lines, use two arrows ($>>$) for the second pair. The two lines to the right would be labeled $\overleftrightarrow{AB} \ || \ \overleftrightarrow{MN}$ or $l \ || \ m$.

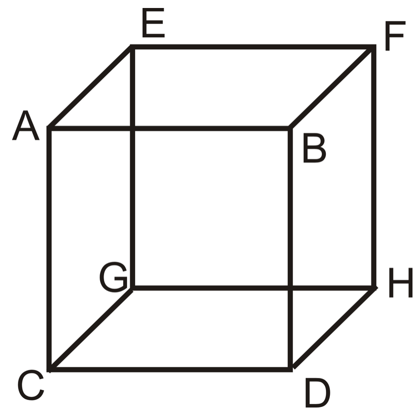


Planes can also be parallel or perpendicular. The image to the left shows two parallel planes, with a third blue plane that is perpendicular to both of them.

An example of parallel planes could be the top of a table and the floor. The legs would be in perpendicular planes to the table top and the floor.

Skew lines: Lines that are in different planes and never intersect.

Example 1: In the cube above, list:



a) 3 pairs of parallel planes

b) 2 pairs of perpendicular planes

c) 3 pairs of skew line segments

Solution:

a) Planes ABC and EFG, Planes AEG and FBH, Planes AEB and CDH

b) Planes ABC and CDH, Planes AEB and FBH (there are others, too)

c) $\overline{BD}$ and $\overline{CG}$, $\overline{BF}$ and $\overline{EG}$, $\overline{GH}$ and $\overline{AE}$ (there are others, too)

Parallel Line Postulate

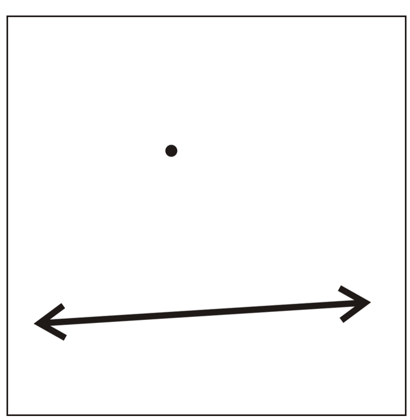
Parallel Line Postulate: Fora line and a point not on the line, there is exactly one line parallel to this line through the point.

There are infinitely many lines that pass through A, but only one is parallel to l.

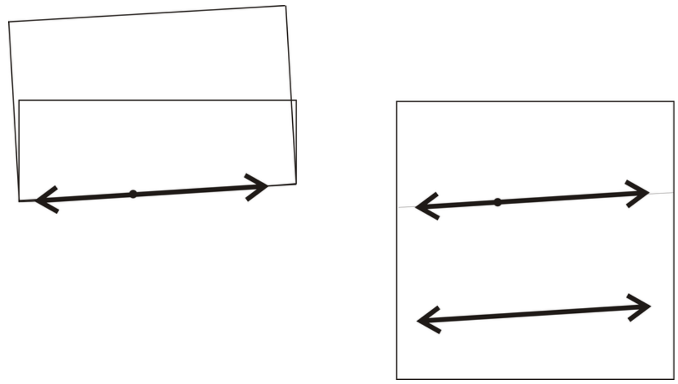
Investigation 3-1: Patty Paper and Parallel Lines

1. Get a piece of patty paper (a translucent square piece of paper).

Draw a line and a point above the line.



2. Fold up the paper so that the line is over the point. Crease the paper and unfold.



3. Are the lines parallel? Yes, by design, this investigation replicates the line we drew in #1 over the point. Therefore, there is only one line parallel through this point to this line.

Perpendicular Line Postulate

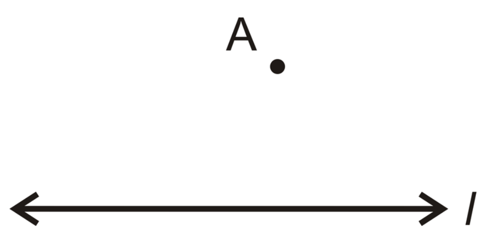
Perpendicular Line Postulate: For a line and a point not on the line, there is exactly one line perpendicular to the line that passes through the point.

There are infinitely many lines that pass through A, but only one that is perpendicular to l.

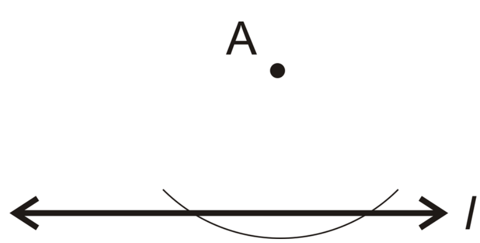
Investigation 3-2: Perpendicular Line Construction; through a Point NOT on the Line

1. Draw a horizontal line and a point above that line.

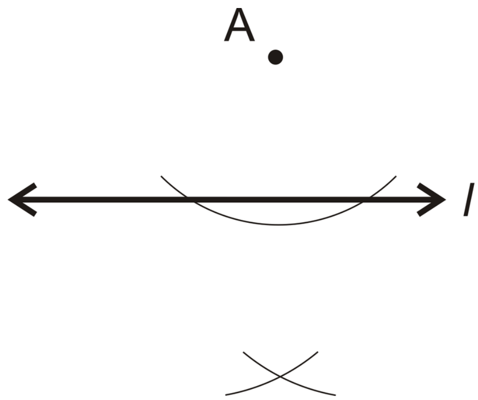
Label the line l and the point A.



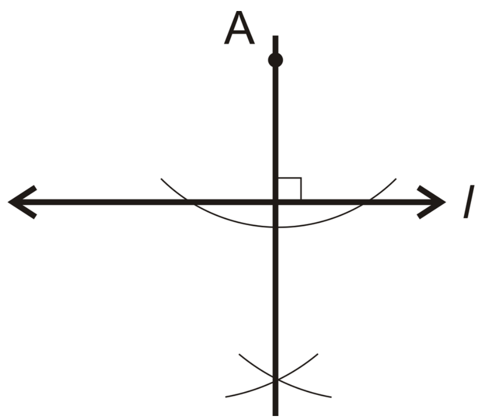
2. Take the compass and put the pointer on A. Open the compass so that it reaches beyond line l. Draw an arc that intersects the line twice.



3. Move the pointer to one of the arc intersections. Widen the compass a little and draw an arc below the line. Repeat this on the other side so that the two arc marks intersect.



4. Take your straightedge and draw a line from point A to the arc intersections below the line. This line is perpendicular to l and passes through A.



Notice that this is a different construction from a perpendicular bisector.

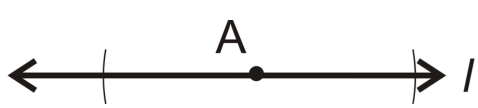
To see a demonstration of this construction, go to: http://www.mathsisfun.com/geometry/construct-perpnotline.html

Investigation 3-3: Perpendicular Line Construction; through a Point on the Line

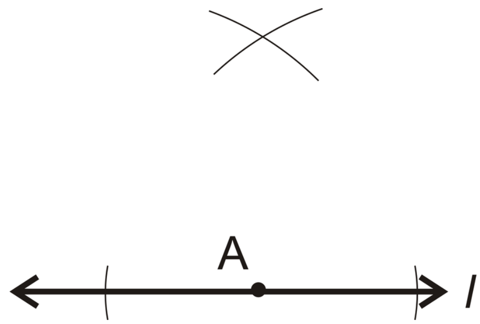
1. Draw a horizontal line and a point on that line.

Label the line l and the point A.

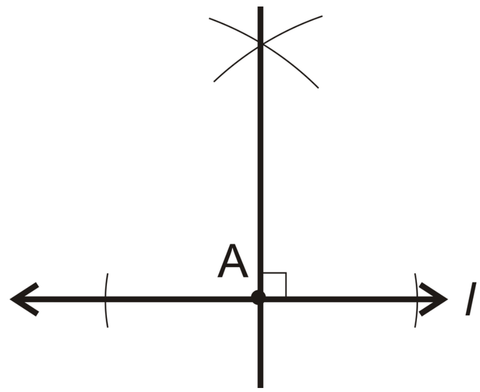
2. Take the compass and put the pointer on A. Open the compass so that it reaches out horizontally along the line. Draw two arcs that intersect the line on either side of the point.



3. Move the pointer to one of the arc intersections. Widen the compass a little and draw an arc above or below the line. Repeat this on the other side so that the two arc marks intersect.



4. Take your straightedge and draw a line from point A to the arc intersections above the line. This line is perpendicular to l and passes through A.



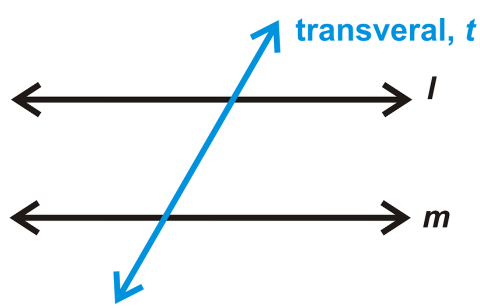
Notice that this is a different construction from a perpendicular bisector.

To see a demonstration of this construction, go to: http://www.mathsisfun.com/geometry/construct-perponline.html

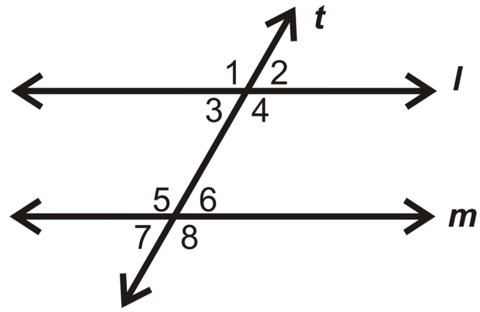
Angles and Transversals

Transversal: A line that intersects two distinct lines. These two lines may or may not be parallel.

The area *between* l and m is the called the *interior*. The area *outside* l and m is called the *exterior*.



Looking at t, l and m, there are 8 angles formed and several linear pairs vertical angle pairs. There are also 4 new angle relationships, defined here:



Corresponding Angles: Two angles that are in the "same place" with respect to the transversal, but on different lines. Imagine sliding the four angles formed with line l down to line m. The angles which match up are corresponding. $\angle 2$ ***and*** $\angle 6$ ***are corresponding angles.***

Alternate Interior Angles: Two angles that are on the interior of l and m, but on opposite sides of the transversal. $\angle 3$ ***and*** $\angle 6$ ***are alternate interior angles.***

Alternate Exterior Angles: Two angles that are on the exterior of l and m, but on opposite sides of the transversal. $\angle 1$ ***and*** $\angle 8$ ***are alternate exterior angles.***

Same Side Interior Angles: Two angles that are on the same side of the transversal and on the interior of the two lines. $\angle 3$ ***and*** $\angle 5$ ***are same side interior angles.***

Example 2: Using the picture above, list all the other pairs of each of the newly defined angle relationships.

Solution:

Corresponding Angles: $\angle 3$ and $\angle 7$, $\angle 1$ and $\angle 5$, $\angle 4$ and $\angle 8$

Alternate Interior Angles: $\angle 4$ and $\angle 5$

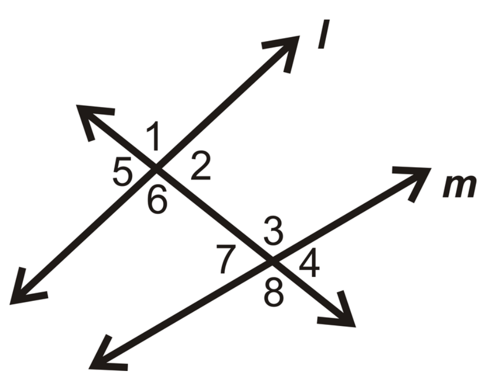
Alternate Exterior Angles: $\angle 2$ and $\angle 7$

Same Side Interior Angles: $\angle 4$ and $\angle 6$

Example 3: If $\angle 2 = 48^\circ$ (in the picture above), what other angles do you know?

Solution: $\angle 2 \cong \angle 3$ by the Vertical Angles Theorem, so $m\angle 3 = 48^\circ$. $\angle 2$ is also a linear pair with $\angle 1$ and $\angle 4$, so it is supplementary to those two. They are both 132°. We do not know the measures of $\angle 5$, $\angle 6$, $\angle 7$, or $\angle 8$ because we do not have enough information.

Example 4: For the picture to the right, determine:



a) A corresponding angle to $\angle 3$?

b) An alternate interior angle to $\angle 7$?

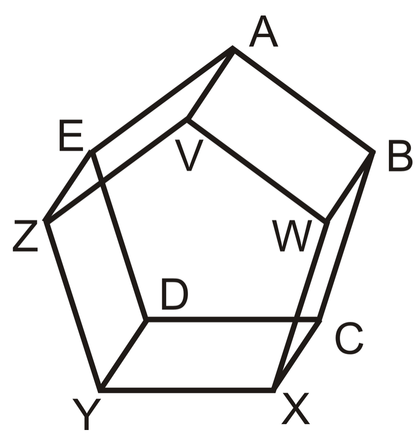
c) An alternate exterior angle to $\angle 4$?

Solution: The corresponding angle to $\angle 3$ is $\angle 1$. The alternate interior angle to $\angle 7$ is $\angle 2$. And, the alternate exterior angle to $\angle 4$ is $\angle 5$.

Know What? Revisited For Washington DC, all of the lettered streets are parallel, as are all of the numbered streets. The lettered streets are perpendicular to the numbered streets. There are no skew streets because all of the streets are in the same plane. We also do not know if any of the state-named streets are parallel or perpendicular.

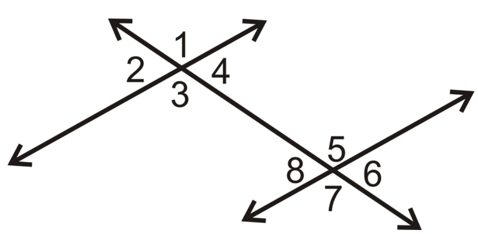
Review Questions

Use the figure below to answer questions 1-5. The two pentagons are parallel and all of the rectangular sides are perpendicular to both of them.



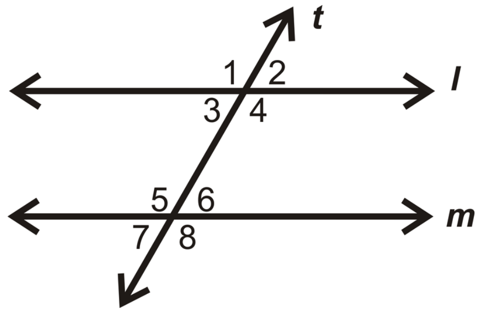
1. Find two pairs of skew lines.
2. List a pair of parallel lines.
3. List a pair of perpendicular lines.
4. For $\overline{AB}$, how many perpendicular lines pass through point V? What line is this?
5. For $\overline{XY}$, how many parallel lines passes through point D? What line is this?

For questions 6-12, use the picture below.



6. What is the corresponding angle to $\angle 4$?
7. What is the alternate interior angle with $\angle 5$?
8. What is the corresponding angle to $\angle 8$?
9. What is the alternate exterior angle with $\angle 7$?
10. What is the alternate interior angle with $\angle 4$?
11. What is the same side interior angle with $\angle 3$?
12. What is the corresponding angle to $\angle 1$?

Use the picture below for questions 13-16.



13. If $m\angle 2 = 55^\circ$, what other angles do you know?
14. If $m\angle 5 = 123^\circ$, what other angles do you know?
15. If $t \perp l$, is $t \perp m$? Why or why not?
16. Is $l \,||\, m$? Why or why not?
17. ***Construction*** Draw a line and a point not on the line. Construct a perpendicular line to your original line through your point.
18. ***Construction*** Construct a perpendicular line to the line you constructed in #12. Use the point you originally drew, so that you will be constructing a perpendicular line through a point on the line.
19. Can you use patty paper to do the construction in number 17? Draw a line and a point not on the line on a piece of patty paper (or any thin white paper or tracing paper). Think about how you could make a crease in the paper that would be a line perpendicular to your original line through your point.
20. Using what you discovered in number 19, use patty paper to construct a line perpendicular to a given line through a point on the given line.
21. Draw a pair of parallel lines using your ruler. Describe how you did this.
22. Draw a pair of perpendicular lines using your ruler. Describe your method.

Geometry is often apparent in nature. Think of examples of each of the following in nature.

23. Parallel Lines or Planes
24. Perpendicular Lines or Planes
25. Skew Lines

Algebra Connection In questions 26-35 we will begin to explore the concepts of parallel and perpendicular lines in the coordinate plane.

26. Write the equations of two lines parallel to $y = 3$.
27. Write the equations of two lines perpendicular to $y = 5$.
28. What is the relationship between the two lines you found for number 27?
29. Plot the points $A(2,-5),\ B(-3,1),\ C(0,4),\ D(-5,10)$. Draw the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. What are the slopes of these lines? What is the geometric relationship between these lines?
30. Plot the points $A(2,1),\ B(7,-2),\ C(2,-2),\ D(5,3)$. Draw the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. What are the slopes of these lines? What is the geometric relationship between these lines?
31. Based on what you discovered in numbers 29 and 30, can you make a conjecture about the slopes of parallel and perpendicular lines?

Find the equation of the line that is <u>parallel</u> to the given line and passes through (5, -1).

32. $y = 2x - 7$
33. $y = -\frac{3}{5}x + 1$

Find the equation of the line that is *<u>perpendicular</u>* to the given line and passes through (2, 3).

34. $y = \frac{2}{3}x - 5$
35. $y = -\frac{1}{4}x + 9$

Review Queue Answers

a. $y = -2x + 3$
b. $y = -4x + 14$
c. $y = \frac{4}{3}x - 4$
d. Yes, the lines are perpendicular. The slopes are reciprocals and opposite signs.

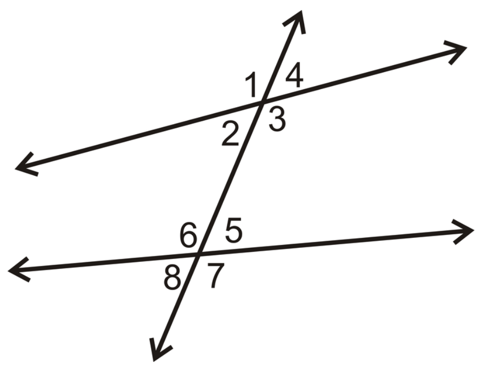
3.2 Properties of Parallel Lines

Learning Objectives

- Use the Corresponding Angles Postulate.
- Use the Alternate Interior Angles Theorem.
- Use the Alternate Exterior Angles Theorem.
- Use Same Side Interior Angles Theorem.

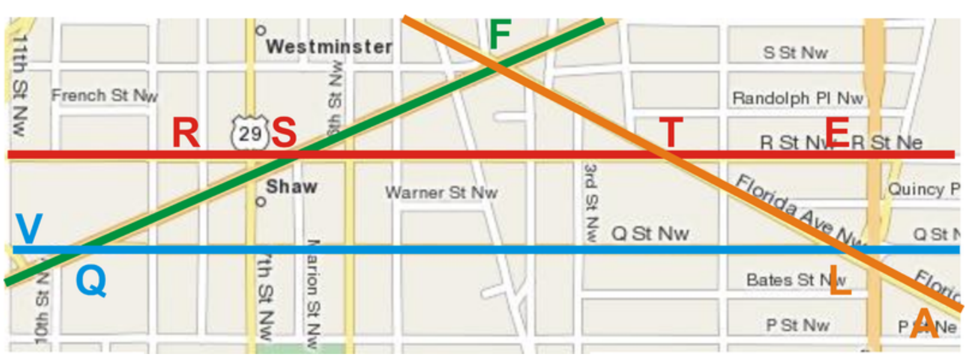
Review Queue

Use the picture below to determine:



a. A pair of corresponding angles.
b. A pair of alternate interior angles.
c. A pair of same side interior angles.
d. If $m\angle 4 = 37^\circ$, what other angles do you know?

Know What? The streets below are in Washington DC. The red street is R St. and the blue street is Q St. These two streets are parallel. The transversals are: Rhode Island Ave. (green) and Florida Ave. (orange).



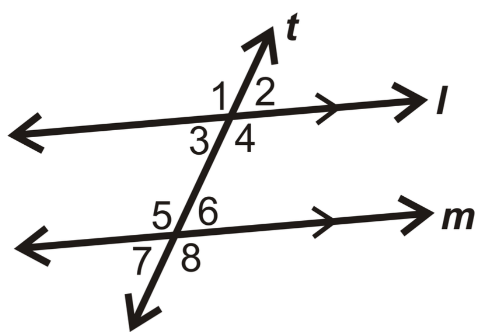
a. If $m\angle FTS = 35^\circ$, determine the other angles that are 35°.
b. If $m\angle SQV = 160^\circ$, determine the other angles that are 160°.
c. Why do you think the "State Streets" exists? Why aren't all the streets parallel or perpendicular?

In this section, we are going to discuss a specific case of two lines cut by a transversal. The two lines are now going to be parallel. If the two lines are parallel, all of the angles, corresponding, alternate interior, alternate exterior and same side interior have new properties. We will begin with corresponding angles.

Corresponding Angles Postulate

Corresponding Angles Postulate: If two parallel lines are cut by a transversal, then the corresponding angles are congruent.

If $l \parallel m$ and both are cut by t, then $\angle 1 \cong \angle 5,\ \angle 2 \cong \angle 6,\ \angle 3 \cong \angle 7$, and $\angle 4 \cong \angle 8$.

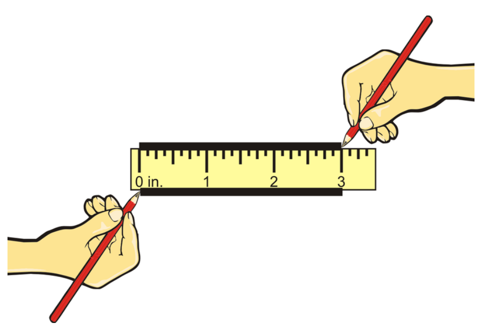


l **must be parallel** to m in order to use this postulate. Recall that a postulate is just like a theorem, but does not need to be proven. We can take it as true and use it just like a theorem from this point.

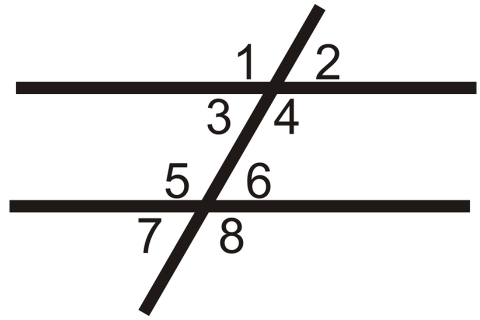
Investigation 3-4: Corresponding Angles Exploration

You will need: paper, ruler, protractor

a. Place your ruler on the paper. On either side of the ruler, draw lines, 3 inches long. This is the easiest way to ensure that the lines are parallel.



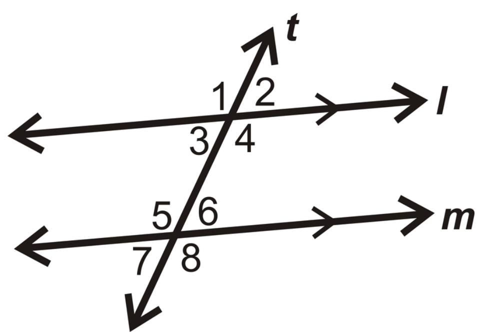
b. Remove the ruler and draw a transversal. Label the eight angles as shown.



c. Using your protractor, measure all of the angles. What do you notice?

In this investigation, you should see that $m\angle 1 = m\angle 4 = m\angle 5 = m\angle 8$ and $m\angle 2 = m\angle 3 = m\angle 6 = m\angle 7$. $\angle 1 \cong \angle 4$, $\angle 5 \cong \angle 8$ by the Vertical Angles Theorem. By the Corresponding Angles Postulate, we can say $\angle 1 \cong \angle 5$ and therefore $\angle 1 \cong \angle 8$ by the Transitive Property. You can use this reasoning for the other set of congruent angles as well.

Example 1: If $m\angle 2 = 76^\circ$, what is $m\angle 6$?



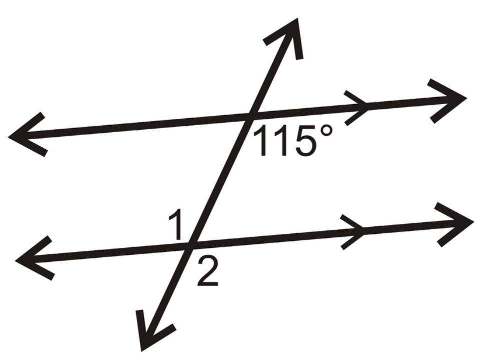
Solution: $\angle 2$ and $\angle 6$ are corresponding angles and $l \,||\, m$, from the markings in the picture. By the Corresponding Angles Postulate the two angles are equal, so $m\angle 6 = 76^\circ$.

Example 2: Using the measures of $\angle 2$ and $\angle 6$ from Example 2, find all the other angle measures.

Solution: If $m\angle 2 = 76^\circ$, then $m\angle 1 = 180^\circ - 76^\circ = 104^\circ$ because they are a linear pair. $\angle 3$ is a vertical angle with $\angle 2$, so $m\angle 3 = 76^\circ$. $\angle 1$ and $\angle 4$ are vertical angles, so $m\angle 4 = 104^\circ$. By the Corresponding Angles Postulate, we know $\angle 1 \cong \angle 5$, $\angle 2 \cong \angle 6$, $\angle 3 \cong \angle 7$, and $\angle 4 \cong \angle 8$, so $m\angle 5 = 104^\circ$, $m\angle 6 = 76^\circ$, $m\angle 7 = 76^\circ$, and $m\angle 104^\circ$.

Alternate Interior Angles Theorem

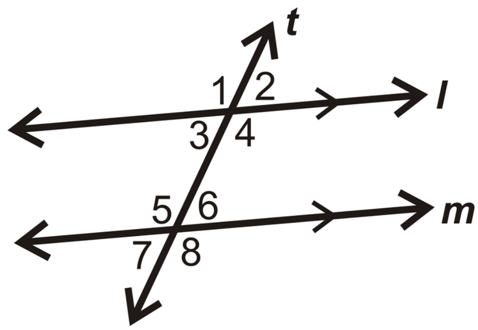
Example 3: Find $m\angle 1$.



Solution: $m\angle 2 = 115^\circ$ because they are corresponding angles and the lines are parallel. $\angle 1$ and $\angle 2$ are vertical angles, so $m\angle 1 = 115^\circ$ also.

$\angle 1$ and the 115° angle are alternate interior angles.

Alternate Interior Angles Theorem: If two parallel lines are cut by a transversal, then the alternate interior angles are congruent.



Proof of Alternate Interior Angles Theorem

Given: $l \parallel m$

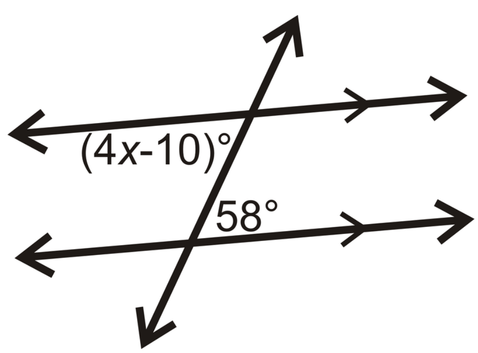
Prove: $\angle 3 \cong \angle 6$

TABLE 3.1:

Statement	***Reason***
1. $l \parallel m$	Given
2. $\angle 3 \cong \angle 7$	Corresponding Angles Postulate
3. $\angle 7 \cong \angle 6$	Vertical Angles Theorem
4. $\angle 3 \cong \angle 6$	Transitive PoC

There are several ways we could have done this proof. For example, Step 2 could have been $\angle 2 \cong \angle 6$ for the same reason, followed by $\angle 2 \cong \angle 3$. We could have also proved that $\angle 4 \cong \angle 5$.

Example 4: ***Algebra Connection*** Find the measure of the angle and x.

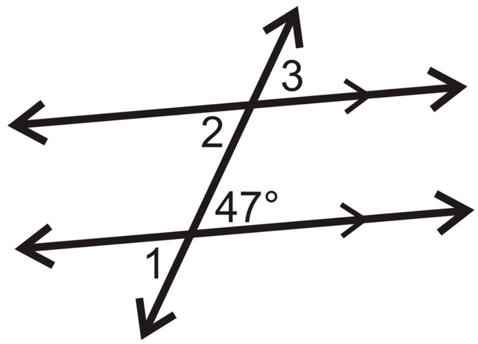


Solution: The two given angles are alternate interior angles so, they are equal. Set the two expressions equal to each other and solve for x.

$$\begin{aligned}(4x-10)^\circ &= 58^\circ\\ 4x &= 68^\circ\\ x &= 17^\circ\end{aligned}$$

Alternate Exterior Angles Theorem

Example 5: Find $m\angle 1$ and $m\angle 3$.



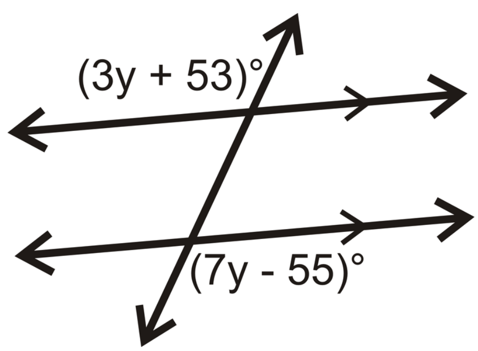
Solution: $m\angle 1 = 47^\circ$ because they are vertical angles. Because the lines are parallel, $m\angle 3 = 47^\circ$ by the Corresponding Angles Theorem. Therefore, $m\angle 2 = 47^\circ$.

$\angle 1$ and $\angle 3$ are alternate exterior angles.

Alternate Exterior Angles Theorem: If two parallel lines are cut by a transversal, then the alternate exterior angles are congruent.

The proof of this theorem is very similar to that of the Alternate Interior Angles Theorem and you will be asked to do in the exercises at the end of this section.

Example 6: ***Algebra Connection*** Find the measure of each angle and the value of y.



Solution: The given angles are alternate exterior angles. Because the lines are parallel, we can set the expressions equal to each other to solve the problem.

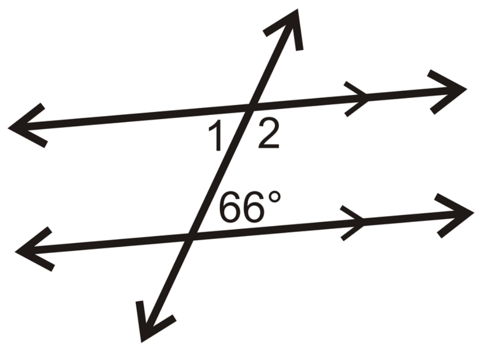
$$\begin{aligned}(3y+53)^\circ &= (7y-55)^\circ\\ 108^\circ &= 4y\\ 27^\circ &= y\end{aligned}$$

If $y = 27^\circ$, then each angle is $3(27^\circ) + 53^\circ$, or 134°.

Same Side Interior Angles Theorem

Same side interior angles have a different relationship that the previously discussed angle pairs.

Example 7: Find $m\angle 2$.



Solution: Here, $m\angle 1 = 66^\circ$ because they are alternate interior angles. $\angle 1$ and $\angle 2$ are a linear pair, so they are supplementary.

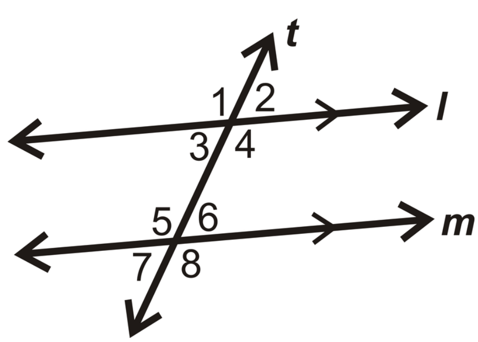
$$\begin{aligned} m\angle 1 + m\angle 2 &= 180^\circ \\ 66^\circ + m\angle 2 &= 180^\circ \\ m\angle 2 &= 114^\circ \end{aligned}$$

This example shows that if two parallel lines are cut by a transversal, the same side interior angles are supplementary.

Same Side Interior Angles Theorem: If two parallel lines are cut by a transversal, then the same side interior angles are supplementary.

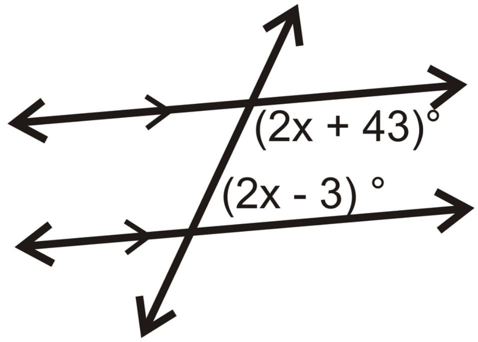
If $l \mid\mid m$ and both are cut by t, then

$m\angle 3 + m\angle 5 = 180^\circ$ and $m\angle 4 + m\angle 6 = 180^\circ$.



You will be asked to do the proof of this theorem in the review questions.

Example 8: ***Algebra Connection*** Find the measure of x.

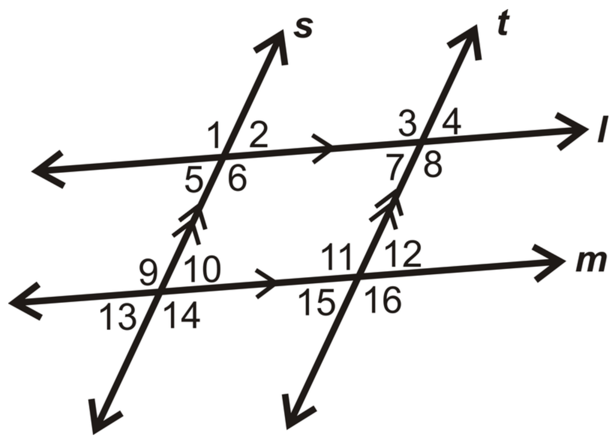


Solution: The given angles are same side interior angles. The lines are parallel, therefore the angles add up to 180°. Write an equation.

$$\begin{aligned}(2x+43)^\circ + (2x-3)^\circ &= 180^\circ \\ (4x+40)^\circ &= 180^\circ \\ 4x &= 140^\circ \\ x &= 35^\circ\end{aligned}$$

While you might notice other angle relationships, there are no more theorems to worry about. However, we will continue to explore these other angle relationships. For example, same side exterior angles are also supplementary. You will prove this in the review questions.

Example 9: $l \,||\, m$ and $s \,||\, t$. Prove $\angle 1 \cong \angle 16$.



Solution:

TABLE 3.2:

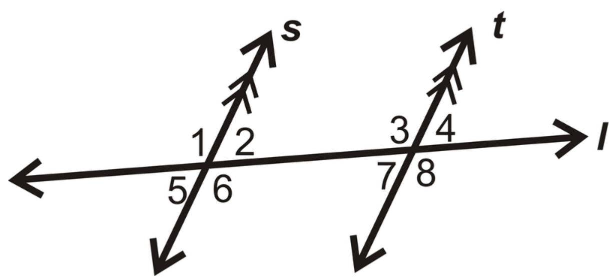
Statement	***Reason***
1. $l \,||\, m$ and $s \,||\, t$	Given
2. $\angle 1 \cong \angle 3$	Corresponding Angles Postulate
3. $\angle 3 \cong \angle 16$	Alternate Exterior Angles Theorem
4. $\angle 1 \cong \angle 16$	Transitive PoC

Know What? Revisited Using what we have learned in this lesson, the other angles that are 35° are $\angle TLQ$, $\angle ETL$, and the vertical angle with $\angle TLQ$. The other angles that are 160° are $\angle FSR$, $\angle TSQ$, and the vertical angle with

$\angle SQV$. You could argue that the "State Streets" exist to help traffic move faster and more efficiently through the city.

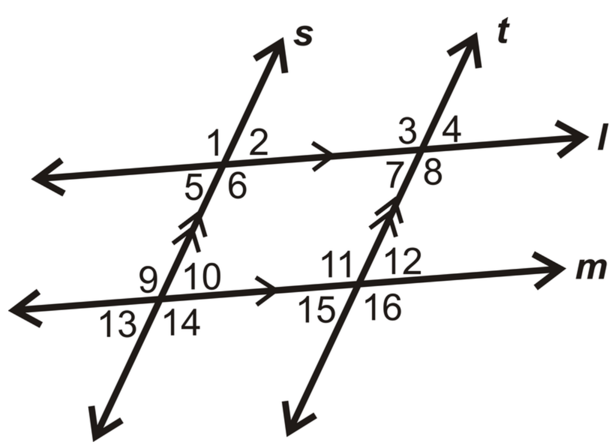
Review Questions

For questions 1-7, determine if each angle pair below is congruent, supplementary or neither.



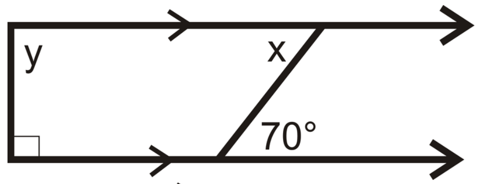
1. $\angle 1$ and $\angle 7$
2. $\angle 4$ and $\angle 2$
3. $\angle 6$ and $\angle 3$
4. $\angle 5$ and $\angle 8$
5. $\angle 1$ and $\angle 6$
6. $\angle 4$ and $\angle 6$
7. $\angle 2$ and $\angle 3$

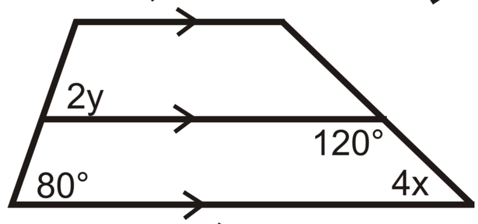
For questions 8-16, determine if the angle pairs below are: Corresponding Angles, Alternate Interior Angles, Alternate Exterior Angles, Same Side Interior Angles, Vertical Angles, Linear Pair or None.

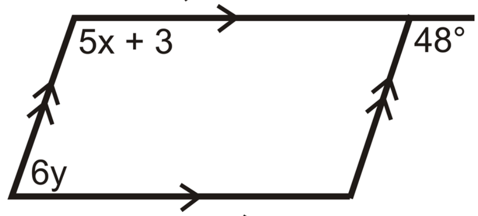


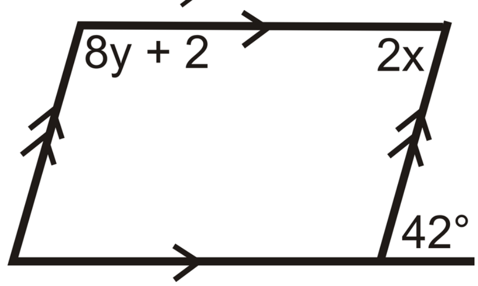
8. $\angle 2$ and $\angle 13$
9. $\angle 7$ and $\angle 12$
10. $\angle 1$ and $\angle 11$
11. $\angle 6$ and $\angle 10$
12. $\angle 14$ and $\angle 9$
13. $\angle 3$ and $\angle 11$
14. $\angle 4$ and $\angle 15$
15. $\angle 5$ and $\angle 16$
16. List all angles congruent to $\angle 8$.

For 17-20, find the values of x and y.

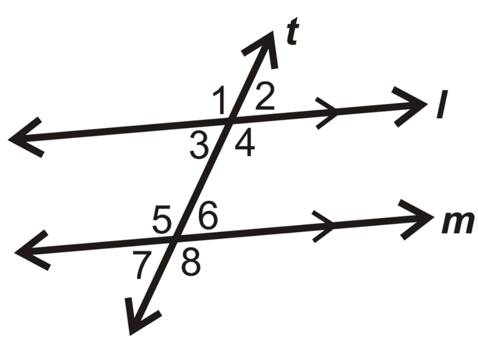
17. 

18. 

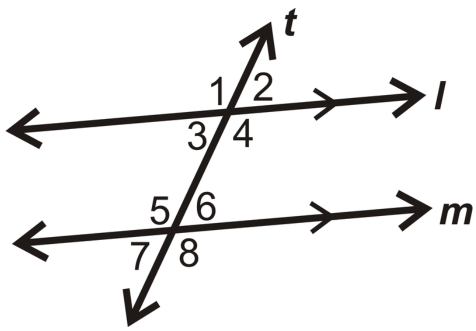
19. 

20. 

Algebra Connection For questions 21-25, use thepicture to the right. Find the value of x and/or y.



21. $m\angle 1 = (4x+35)^\circ$, $m\angle 8 = (7x-40)^\circ$
22. $m\angle 2 = (3y+14)^\circ$, $m\angle 6 = (8x-76)^\circ$
23. $m\angle 3 = (3x+12)^\circ$, $m\angle 5 = (5x+8)^\circ$
24. $m\angle 4 = (5x-33)^\circ$, $m\angle 5 = (2x+60)^\circ$
25. $m\angle 1 = (11y-15)^\circ$, $m\angle 7 = (5y+3)^\circ$
26. Fill in the blanks in the proof below.

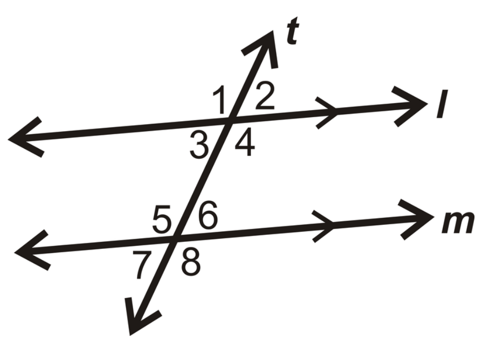


Given: $l \parallel m$Prove: $\angle 3$ and $\angle 5$ are supplementary (Same Side Interior Angles Theorem)

TABLE 3.3:

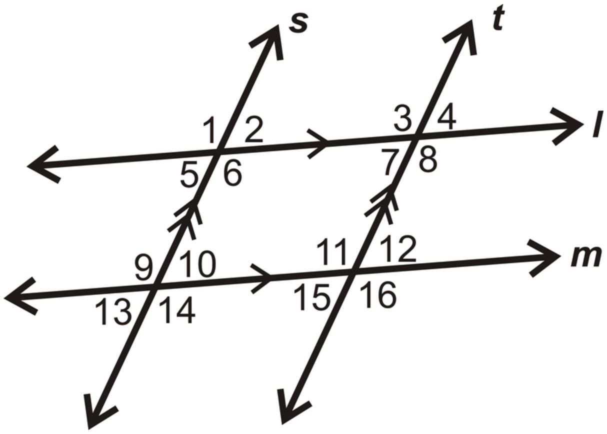
Statement	Reason
1.	Given
2. $\angle 1 \cong \angle 5$	
3.	$\cong$ angles have = measures
4.	Linear Pair Postulate
5.	Definition of Supplementary Angles
6. $m\angle 3 + m\angle 5 = 180^\circ$	
7. $\angle 3$ and $\angle 5$ are supplementary	

For 27 and 28, use the picture to the right to complete each proof.

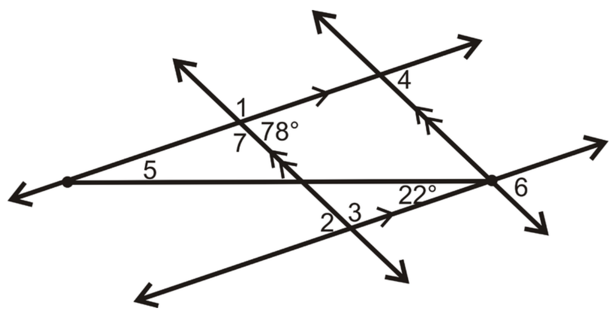


27. Given: $l \parallel m$Prove: $\angle 1 \cong \angle 8$ (Alternate Exterior Angles Theorem)
28. Given: $l \parallel m$Prove: $\angle 2$ and $\angle 8$ are supplementary

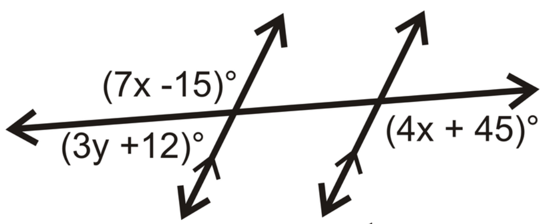
For 29-31, use the picture to the right to complete each proof.



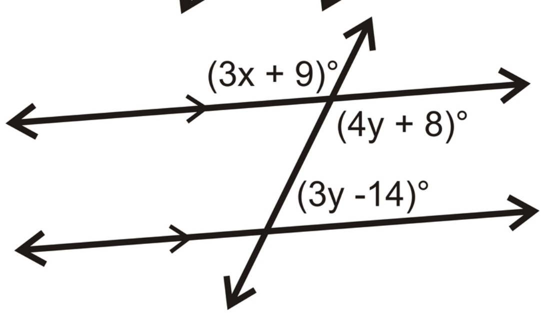
29. Given: $l \| m$, $s \| t$Prove: $\angle 4 \cong \angle 10$
30. Given: $l \| m$, $s \| t$Prove: $\angle 2 \cong \angle 15$
31. Given: $l \| m$, $s \| t$Prove: $\angle 4$ and $\angle 9$ are supplementary
32. Find the measures of all the numbered angles in the figure below.



Algebra Connection For 32 and 33, find the values of x and y.



33.



34.

35. ***Error Analysis*** Nadia is working on Problem 31. Here is her proof:

TABLE 3.4:

Statement	***Reason***
1. $l \| m$, $s \| t$	Given
2. $\angle 4 \cong \angle 15$	Alternate Exterior Angles Theorem

TABLE 3.4: (continued)

Statement	***Reason***
3. $\angle 15 \cong \angle 14$	Same Side Interior Angles Theorem
4. $\angle 14 \cong \angle 9$	Vertical Angles Theorem
5. $\angle 4 \cong \angle 9$	Transitive PoC

What happened? Explain what is needed to be done to make the proof correct.

Review Queue Answers

a. $\angle 1$ and $\angle 6, \angle 2$ and $\angle 8, \angle 3$ and $\angle 7$, or $\angle 4$ and $\angle 5$
b. $\angle 2$ and $\angle 5$ or $\angle 3$ and $\angle 6$
c. $\angle 1$ and $\angle 7$ or $\angle 4$ and $\angle 8$
d. $\angle 3$ and $\angle 5$ or $\angle 2$ and $\angle 6$

3.3 Proving Lines Parallel

Learning Objectives

- Use the *converses* of the Corresponding Angles Postulate, Alternate Interior Angles Theorem, Alternate Exterior Angles Theorem, and the Same Side Interior Angles Theorem to show that lines are parallel.
- Construct parallel lines using the above converses.
- Use the Parallel Lines Property.

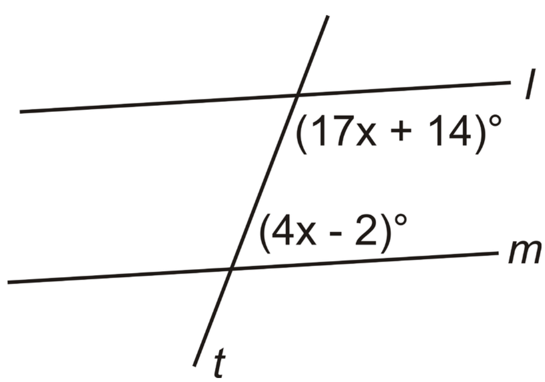
Review Queue

Answer the following questions.

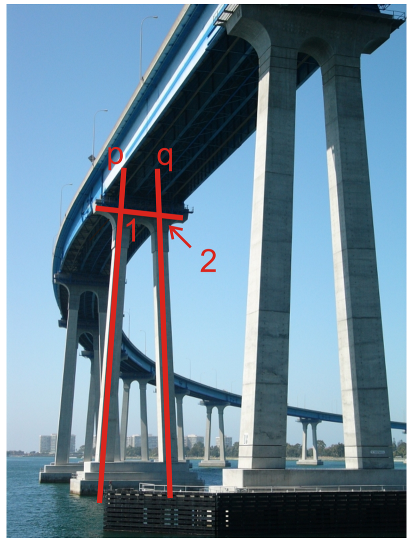
a. Write the converse of the following statements:

   a. *If it is summer, then I am out of school.*
   b. *I will go to the mall when I am done with my homework.*
   c. *If two parallel lines are cut by a transversal, then the corresponding angles are congruent.*

b. Are any of the three converses from #1 true? Why or why not? Give a counterexample.
c. Determine the value of x if $l \parallel m$.



Know What? Here is a picture of the support beams for the Coronado Bridge in San Diego. This particular bridge, called a girder bridge, is usually used in straight, horizontal situations. The Coronado Bridge is diagonal, so the beams are subject to twisting forces (called torque). This can be fixed by building a curved bridge deck. To aid the curved bridge deck, the support beams should not be parallel. If they are, the bridge would be too fragile and susceptible to damage.



This bridge was designed so that $\angle 1 = 92^\circ$ and $\angle 2 = 88^\circ$. Are the support beams parallel?

Corresponding Angles Converse

Recall that the converse of a statement switches the conclusion and the hypothesis. So, if a, then b becomes if b, then a. We will find the converse of all the theorems from the last section and will determine if they are true.

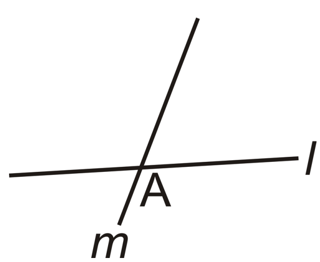
The Corresponding Angles Postulate says: *If two lines are parallel, then the corresponding angles are congruent.* The converse is:

Converse of Corresponding Angles Postulate: If corresponding angles are congruent when two lines are cut by a transversal, then the lines are parallel.

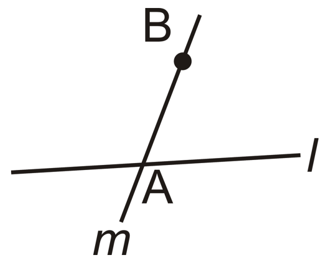
Is this true? For example, if the corresponding angles both measured 60°, would the lines be parallel? YES. All eight angles created by l, m and the transversal are either 60° or 120°, making the slopes of l and m the same which makes them parallel. This can also be seen by using a construction.

Investigation 3-5: Creating Parallel Lines using Corresponding Angles

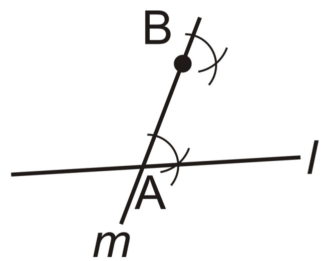
a. Draw two intersecting lines. Make sure they are not perpendicular. Label them l and m, and the point of intersection, A, as shown.



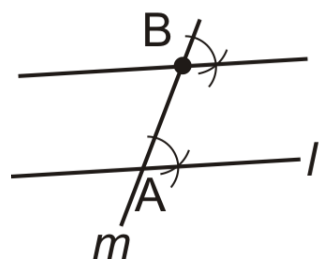
b. Create a point, B, on line m, above A.



c. Copy the acute angle at A (the angle to the right of m) at point B. See Investigation 2-2 in Chapter 2 for the directions on how to copy an angle.

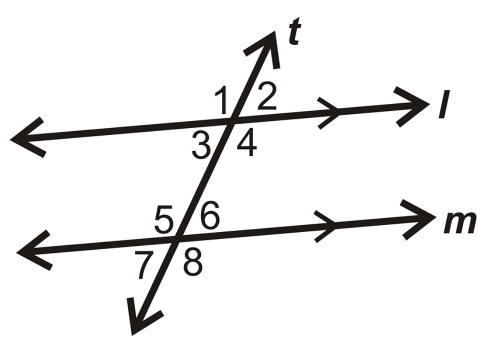


d. Draw the line from the arc intersections to point B.



From this construction, we can see that the lines are parallel.

Example 1: If $m\angle 8 = 110^\circ$ and $m\angle 4 = 110^\circ$, then what do we know about lines l and m?



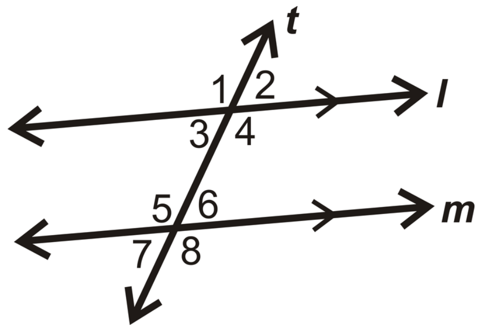
Solution: $\angle 8$ and $\angle 4$ are corresponding angles. Since $m\angle 8 = m\angle 4$, we can conclude that $l \mid\mid m$.

Alternate Interior Angles Converse

We also know, from the last lesson, that when parallel lines are cut by a transversal, the alternate interior angles are congruent. The converse of this theorem is also true:

Converse of Alternate Interior Angles Theorem: If two lines are cut by a transversal and alternate interior angles are congruent, then the lines are parallel.

Example 3: Prove the Converse of the Alternate Interior Angles Theorem.



Given: l and m and transversal t

$\angle 3 \cong \angle 6$

Prove: $l \,||\, m$

Solution:

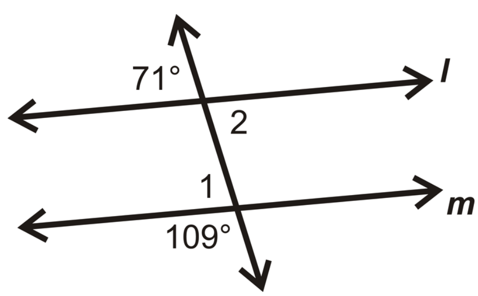
TABLE 3.5:

Statement	*Reason*
1. l and m and transversal $t \angle 3 \cong \angle 6$	Given
2. $\angle 3 \cong \angle 2$	Vertical Angles Theorem
3. $\angle 2 \cong \angle 6$	Transitive PoC
4. $l \,||\, m$	Converse of the Corresponding Angles Postulate

Prove Move: Shorten the names of these theorems. Discuss with your teacher an appropriate abbreviations. For example, the Converse of the Corresponding Angles Theorem could be "Converse CA Thm" or "ConvCA."

Notice that the Corresponding Angles Postulate was not used in this proof. The Transitive Property is the reason for Step 3 because we do not know if l is parallel to m until we are done with the proof. You could conclude that if we are trying to prove two lines are parallel, the converse theorems will be used. And, if we are proving two angles are congruent, we must be given that the two lines are parallel.

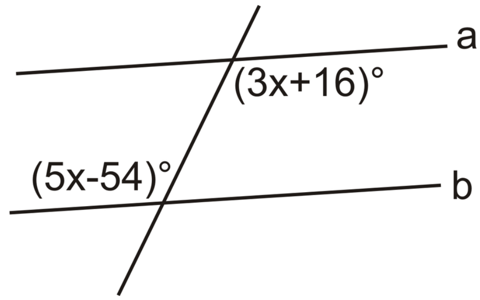
Example 4: Is $l \,||\, m$?



Solution: First, find $m\angle 1$. We know its linear pair is 109°. By the Linear Pair Postulate, these two angles add up to 180°, so $m\angle 1 = 180^\circ - 109^\circ = 71^\circ$. This means that $l \,||\, m$, by the Converse of the Corresponding Angles Postulate.

Example 5: ***Algebra Connection*** What does x have to be to make $a \,||\, b$?

Solution: Because these are alternate interior angles, they must be equal for $a \,||\, b$. Set the expressions equal to each other and solve.



$$
\begin{aligned}
3x+16^\circ &= 5x-54^\circ \\
70^\circ &= 2x \\
35^\circ &= x \qquad \text{To make } a \parallel b,\ x=35^\circ.
\end{aligned}
$$

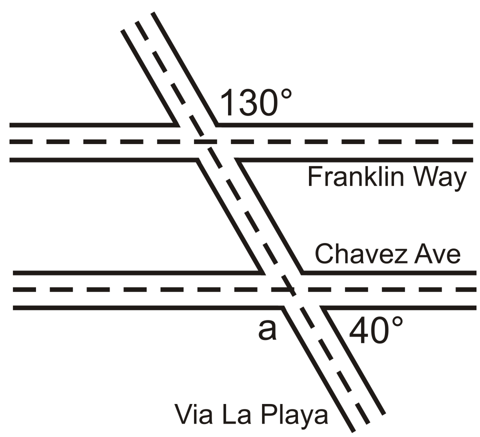
Converse of Alternate Exterior Angles & Consecutive Interior Angles

You have probably guessed that the converse of the Alternate Exterior Angles Theorem and the Consecutive Interior Angles Theorem areal so true.

Converse of the Alternate Exterior Angles Theorem: If two lines are cut by a transversal and the alternate exterior angles are congruent, then the lines are parallel.

Example 6: ***Real-World Situation*** The map below shows three roads in Julio's town.

Julio used a surveying tool to measure two angles at the intersections in this picture he drew (NOT to scale). ***Julio wants to know if Franklin Way is parallel to Chavez Avenue.***



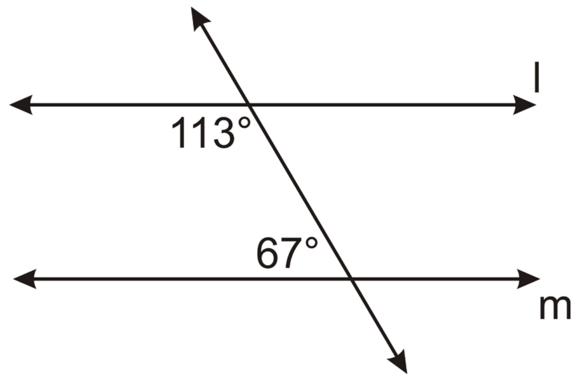
Solution: The labeled 130° angle and $\angle a$ are alternate exterior angles. If $m\angle a = 130^\circ$, then the lines are parallel. To find $m\angle a$, use the other labeled angle which is 40°, and its linear pair. Therefore, $\angle a + 40^\circ = 180^\circ$ and $\angle a = 140^\circ$. $140^\circ \neq 130^\circ$, so Franklin Way and Chavez Avenue are not parallel streets.

The final converse theorem is of the Same Side Interior Angles Theorem. Remember that these angles are not congruent when lines are parallel, they are **supplementary**.

Converse of the Same Side Interior Angles Theorem: If two lines are cut by a transversal and the consecutive interior angles are supplementary, then the lines are parallel.

Example 7: Is $l \parallel m$? How do you know?

Solution: These are Same Side Interior Angles. So, if they add up to 180°, then $l \,||\, m$. $113^\circ + 67^\circ = 180^\circ$, therefore $l \,||\, m$.

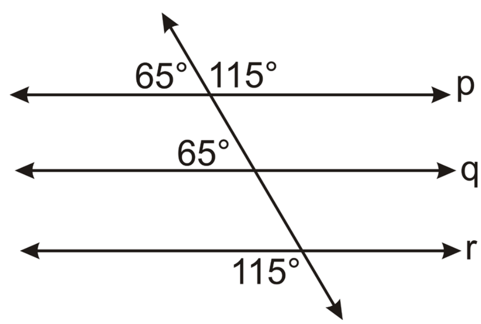


Parallel Lines Property

The Parallel Lines Property is a transitive property that can be applied to parallel lines. Remember the Transitive Property of Equality is: If $a = b$ and $b = c$, then $a = c$. The Parallel Lines Property changes = to ||.

Parallel Lines Property: If lines $l \,||\, m$ and $m \,||\, n$, then $l \,||\, n$.

Example 8: Are lines q and r parallel?



Solution: First find if $p \,||\, q$, followed by $p \,||\, r$. If so, then $q \,||\, r$.

$p \,||\, q$ by the Converse of the Corresponding Angles Postulate, the corresponding angles are 65°. $p \,||\, r$ by the Converse of the Alternate Exterior Angles Theorem, the alternate exterior angles are 115°. Therefore, by the Parallel Lines Property, $q \,||\, r$.

Know What? Revisited: The CoronadoBridge has $\angle 1$ and $\angle 2$, which are corresponding angles. These angles must be equal for the beams to be parallel. $\angle 1 = 92^\circ$ and $\angle 2 = 88^\circ$ and $92^\circ \neq 88^\circ$, so the beams are not parallel, therefore a sturdy and safe girder bridge.

Review Questions

1. ***Construction*** Using Investigation 3-1 to help you, show that two lines are parallel by constructing congruent alternate interior angles. HINT: Steps 1 and 2 will be exactly the same, but at step 3, you will copy the angle in a different location.

2. ***Construction*** Using Investigation 3-1 to help you, show that two lines are parallel by constructing supplementary consecutive interior angles. HINT: Steps 1 and 2 will be exactly the same, but at step 3, you will copy a different angle.

For Questions 3-5, fill in the blanks in the proofs below.

3. Given: $l \parallel m,\ p \parallel q$ Prove: $\angle 1 \cong \angle 2$

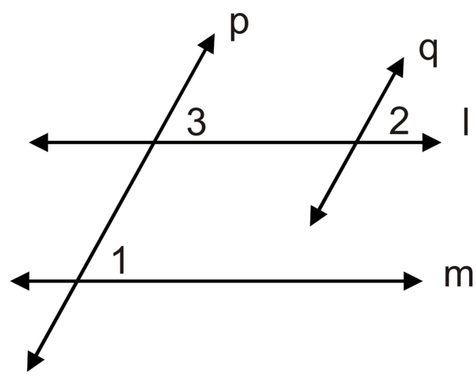


TABLE 3.6:

Statement	*Reason*
1. $l \parallel m$	1.
2.	2. Corresponding Angles Postulate
3. $p \parallel q$	3.
4.	4.
5. $\angle 1 \cong \angle 2$	5.

4. Given: $p \parallel q,\ \angle 1 \cong \angle 2$ Prove: $l \parallel m$

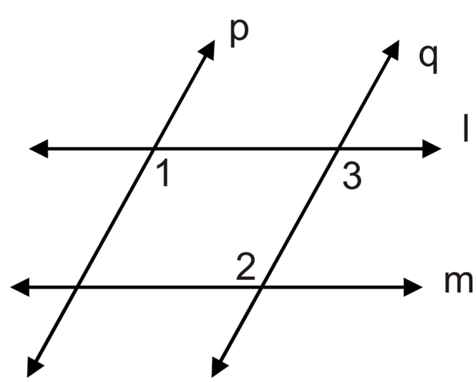


TABLE 3.7:

Statement	*Reason*
1. $p \parallel q$	1.
2.	2. Corresponding Angles Postulate
3. $\angle 1 \cong \angle 2$	3.
4.	4. Transitive PoC
5.	5. Converse of Alternate Interior Angles Theorem

5. Given: $\angle 1 \cong \angle 2,\ \angle 3 \cong \angle 4$Prove: $l \,||\, m$

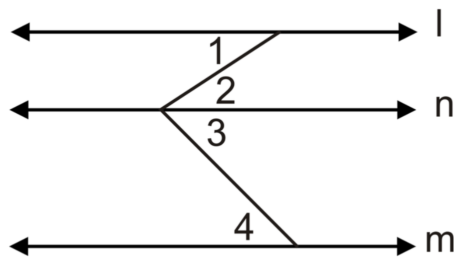
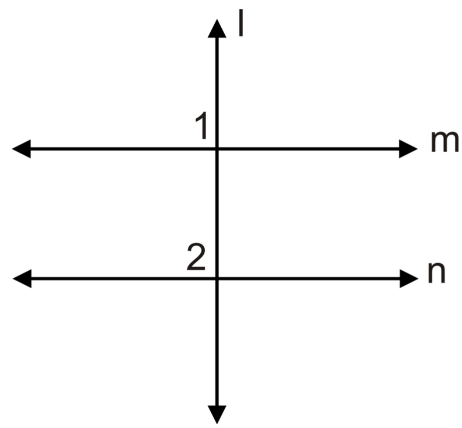


TABLE 3.8:

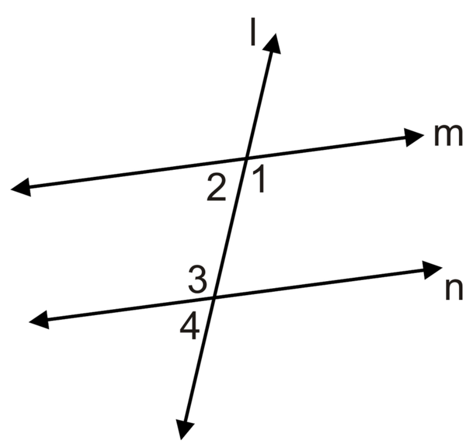
Statement	***Reason***
1. $\angle 1 \cong \angle 2$	1.
2. $l \,||\, n$	2.
3. $\angle 3 \cong \angle 4$	3.
4.	4. Converse of Alternate Interior Angles Theorem
5. $l \,||\, m$	5.

For Questions 6-9, create your own two column proof.

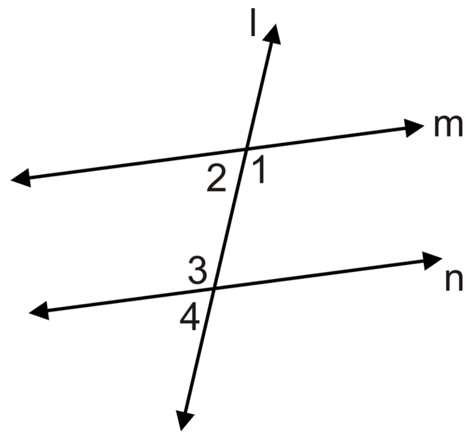
6. Given: $m \perp l,\ n \perp l$Prove: $m \,||\, n$



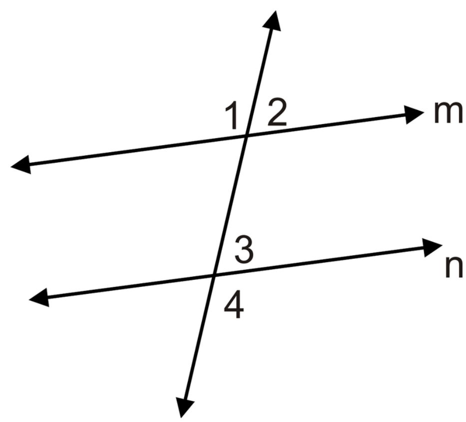
7. Given: $\angle 1 \cong \angle 3$Prove: $\angle 1$ and $\angle 4$ are supplementary



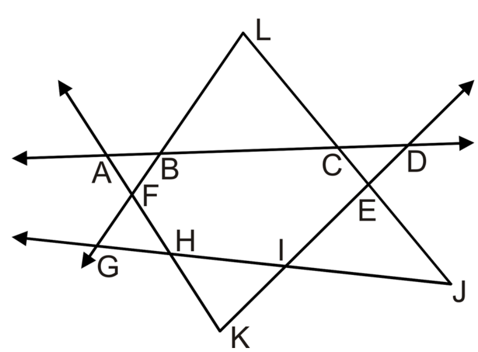
8. Given: $\angle 2 \cong \angle 4$Prove: $\angle 1 \cong \angle 3$



9. Given: $\angle 2 \cong \angle 3$Prove: $\angle 1 \cong \angle 4$

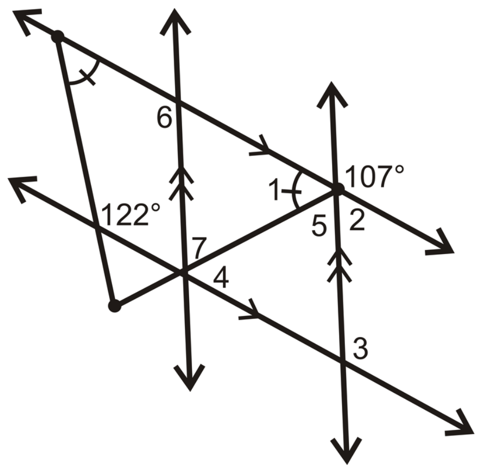


In 10-15, use the given information to determine which lines are parallel. If there are none, write *none*. Consider each question individually.



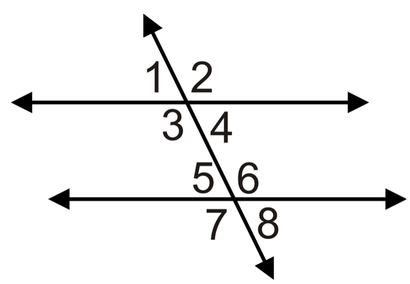
10. $\angle LCD \cong \angle CJI$
11. $\angle BCE$ *and* $\angle BAF$ are supplementary
12. $\angle FGH \cong \angle EIJ$
13. $\angle BFH \cong \angle CEI$
14. $\angle LBA \cong \angle IHK$
15. $\angle ABG \cong \angle BGH$

In 16-22, find the measure of the lettered angles below.

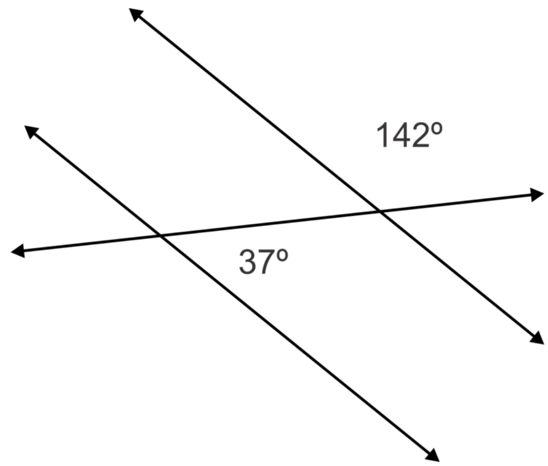


16. $m\angle 1$
17. $m\angle 2$
18. $m\angle 3$
19. $m\angle 4$
20. $m\angle 5$
21. $m\angle 6$
22. $m\angle 7$

For 23-27, what does x have to measure to make the lines parallel?



23. $m\angle 3 = (3x+25)^\circ$ and $m\angle 5 = (4x-55)^\circ$
24. $m\angle 2 = (8x)^\circ$ and $m\angle 7 = (11x-36)^\circ$
25. $m\angle 1 = (6x-5)^\circ$ and $m\angle 5 = (5x+7)^\circ$
26. $m\angle 4 = (3x-7)^\circ$ and $m\angle 7 = (5x-21)^\circ$
27. $m\angle 1 = (9x)^\circ$ and $m\angle 6 = (37x)^\circ$
28. ***Construction*** Draw a straight line. Construct a line perpendicular to this line through a point on the line. Now, construct a perpendicular line to this new line. What can you conclude about the original line and this final line?
29. How could you prove your conjecture from problem 28?
30. What is wrong in the following diagram, given that $j \,||\, k$?



Review Queue Answers

a.
   a. *If I am out of school, then it is summer.*
   b. *If I go to the mall, then I am done with my homework.*
   c. *If corresponding angles created by two lines cut by a transversal are congruent, then the two lines are parallel.*

   a. Not true, I could be out of school on any school holiday or weekend during the school year.
   b. Not true, I don't have to be done with my homework to go to the mall.
   c. Yes, because if two corresponding angles are congruent, then the slopes of these two lines have to be the same, making the lines parallel.

b. The two angles are supplementary.

$$\begin{aligned}(17x+14)^\circ+(4x-2)^\circ &= 180^\circ\\ 21x+12^\circ &= 180^\circ\\ 21x &= 168^\circ\\ x &= 8^\circ\end{aligned}$$

3.4 Properties of Perpendicular Lines

Learning Objectives

- Understand the properties of perpendicular lines.
- Explore problems with parallel lines and a perpendicular transversal.
- Solve problems involving complementary adjacent angles.

Review Queue

Determine if the following statements are true or false. If they are true, write the converse. If they are false, find a counter example.

1. Perpendicular lines form four right angles.

2. A right angle is greater than or equal to 90°.

Find the slope between the two given points.

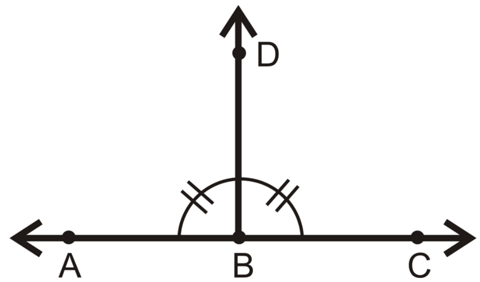
3. (-3, 4) and (-3, 1)

4. (6, 7) and (-5, 7)

Know What? There are several examples of slope in nature. To the right are pictures of Half Dome, in Yosemite-National Park and the horizon over the Pacific Ocean. These are examples of horizontal and vertical lines in real life. Can you determine the slope of these lines?

Congruent Linear Pairs

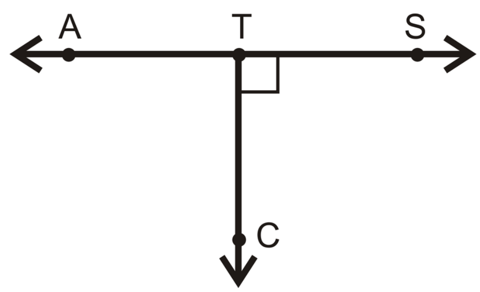
Recall that a linear pair is a pair of adjacent angles whose outer sides form a straight line. The Linear Pair Postulate says that the angles in a linear pair are supplementary. What happens when the angles in a linear pair are congruent?



$$
\begin{aligned}
m\angle ABD + m\angle DBC &= 180^\circ && \text{Linear Pair Postulate} \\
m\angle ABD &= m\angle DBC && \text{The two angles are congruent} \\
m\angle ABD + m\angle ABD &= 180^\circ && \text{Substitution PoE} \\
2m\angle ABD &= 180^\circ && \text{Combine like terms} \\
m\angle ABD &= 90^\circ && \text{Division PoE}
\end{aligned}
$$

So, anytime a linear pair is congruent, the angles are both 90°.

Example 1: Find $m\angle CTA$.

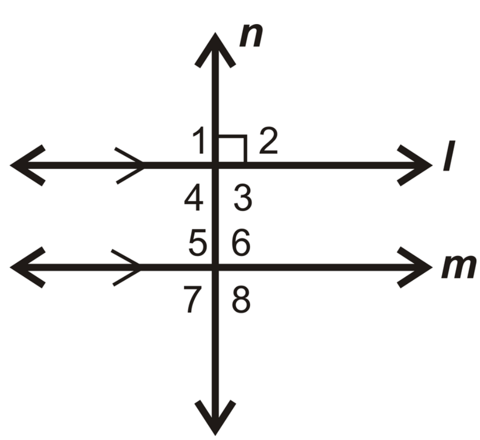


Solution: First, these two angles form a linear pair. Second, from the marking, we know that $\angle STC$ is a right angle. Therefore, $m\angle STC = 90^\circ$. So, $m\angle CTA$ is also 90°.

Perpendicular Transversals

Recall that when two lines intersect, four angles are created. If the two lines are perpendicular, then all four angles are right angles, even though only one needs to be marked with the square. Therefore, all four angles are 90°.

When a parallel line is added, then there are eight angles formed. If $l \parallel m$ and $n \perp l$, is $n \perp m$? Let's prove it here.



Given: $l \parallel m$, $l \perp n$

Prove: $n \perp m$

TABLE 3.9:

Statement	*Reason*
1. $l \parallel m$, $l \perp n$	Given
2. $\angle 1$, $\angle 2$, $\angle 3$, and $\angle 4$ are right angles	Definition of perpendicular lines
3. $m\angle 1 = 90^\circ$	Definition of a right angle
4. $m\angle 1 = m\angle 5$	Corresponding Angles Postulate
5. $m\angle 5 = 90^\circ$	Transitive PoE
6. $m\angle 6 = m\angle 7 = 90^\circ$	Congruent Linear Pairs
7. $m\angle 8 = 90^\circ$	Vertical Angles Theorem
8. $\angle 5$, $\angle 6$, $\angle 7$, and $\angle 8$ are right angles	Definition of right angle
9. $n \perp m$	Definition of perpendicular lines

Theorem 3-1: If two lines are parallel and a third line is perpendicular to one of the parallel lines, it is also perpendicular to the other parallel line.

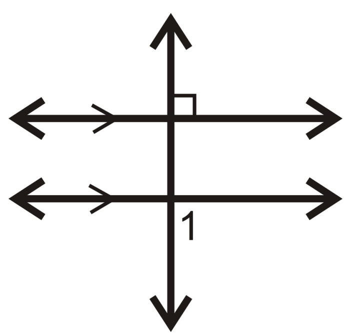
Or, if $l \parallel m$ and $l \perp n$, then $n \perp m$.

Theorem 3-2: If two lines are perpendicular to the same line, they are parallel to each other.

Or, if $l \perp n$ and $n \perp m$, then $l \parallel m$. You will prove this theorem in the review questions.

From these two theorems, we can now assume that any angle formed by two parallel lines and a perpendicular transversal will always be 90°.

Example 2: Determine the measure of $\angle 1$.



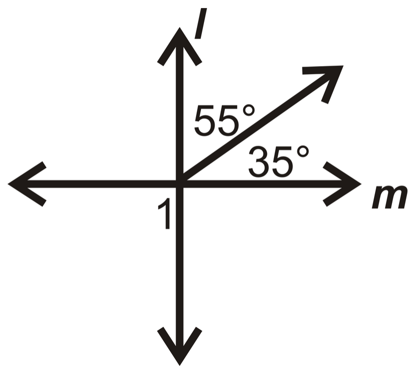
Solution: From Theorem 3-1, we know that the lower parallel line is also perpendicular to the transversal. Therefore, $m\angle 1 = 90^\circ$.

Adjacent Complementary Angles

Recall that complementary angles add up to 90°. If complementary angles are adjacent, their nonadjacent sides are perpendicular rays. What you have learned about perpendicular lines can be applied to this situation.

Example 3: Find $m\angle 1$.

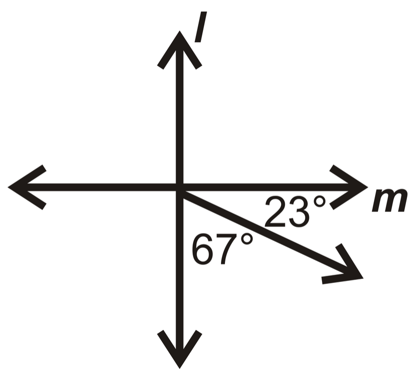
Solution: The two adjacent angles add up to 90°, so $l \perp m$. Therefore, $m\angle 1 = 90^\circ$.



Example 4: Is $l \perp m$? Explain why or why not.

Solution: If the two adjacent angles add up to 90°, then l and m are perpendicular.

$23^\circ + 67^\circ = 90^\circ$. Therefore, $l \perp m$.



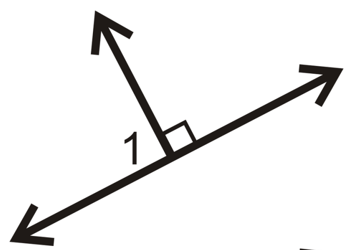
Know What? Revisited

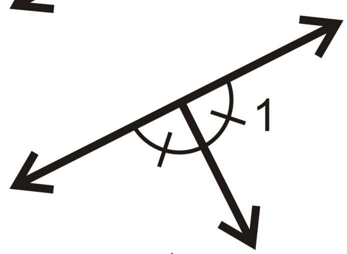
Half Dome is vertical and the slope of any vertical line is undefined. Thousands of people flock to Half Dome to attempt to scale the rock. This front side is very difficult to climb because it is vertical. The only way to scale the front side is to use the provided cables at the base of the rock. http://www.nps.gov/yose/index.htm

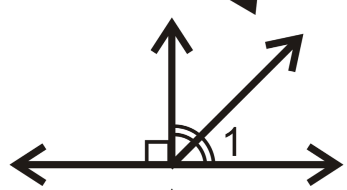
Any horizon over an ocean is horizontal, which has a slope of zero, or no slope. There is no steepness, so no incline or decline. The complete opposite of Half Dome. Actually, if Half Dome was placed on top of an ocean or flat ground, the two would be perpendicular!

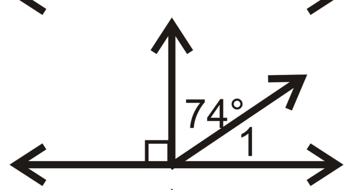
Review Questions

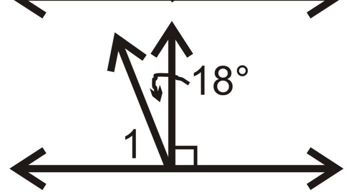
Find the measure of $\angle 1$ for each problem below.

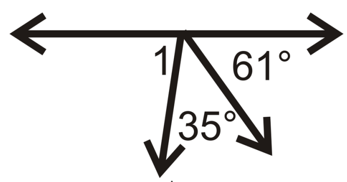
1. 

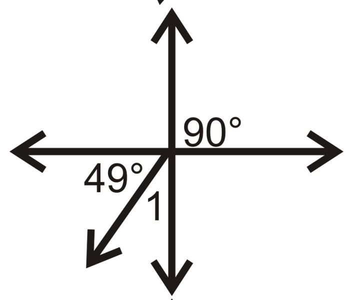
2. 

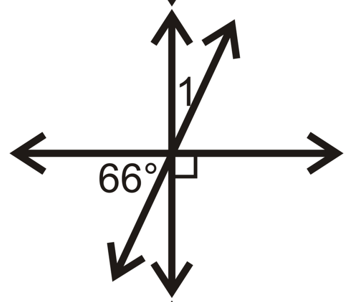
3. 

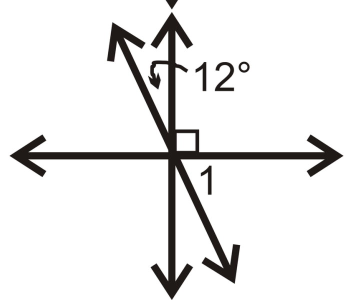
4. 

5. 

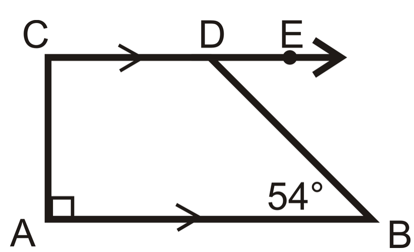
6. 

7. 

8. 

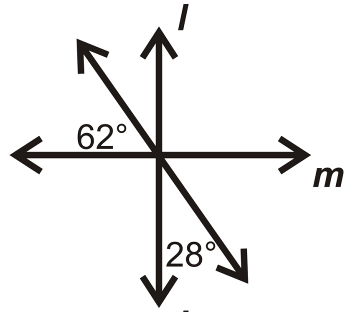
9. 

For questions 10-13, use the picture below.

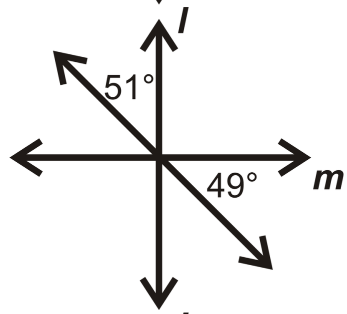


10. Find $m\angle ACD$.
11. Find $m\angle CDB$.
12. Find $m\angle EDB$.
13. Find $m\angle CDE$.

In questions 14-17, determine if $l \perp m$.



14.



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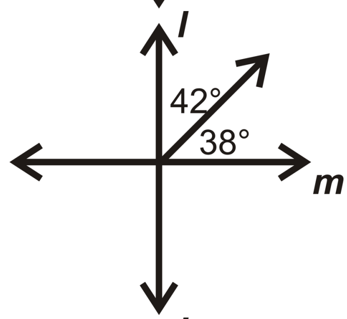
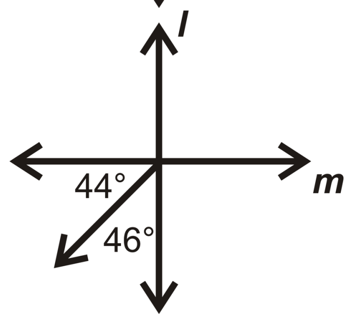
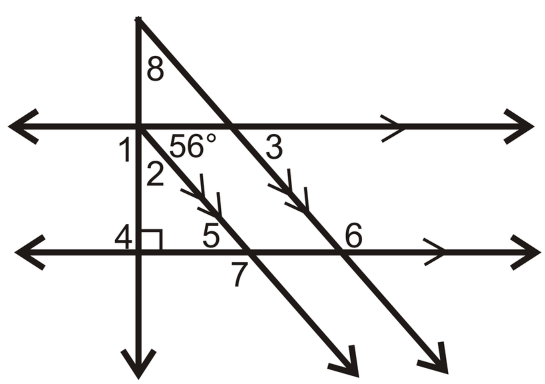


16.



17.

For questions 18-25, use the picture below.

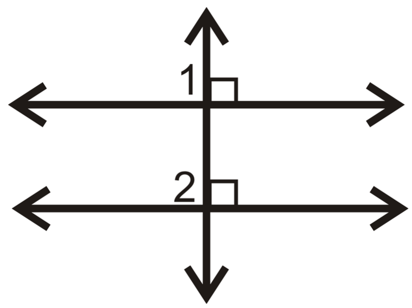


18. Find $m\angle 1$.
19. Find $m\angle 2$.
20. Find $m\angle 3$.

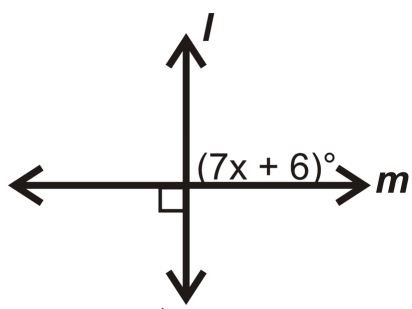
21. Find $m\angle 4$.
22. Find $m\angle 5$.
23. Find $m\angle 6$.
24. Find $m\angle 7$.
25. Find $m\angle 8$.

Complete the proof.

26. Given: $l \perp m$, $l \perp n$Prove: $m \parallel n$



Algebra Connection Find the value of x.



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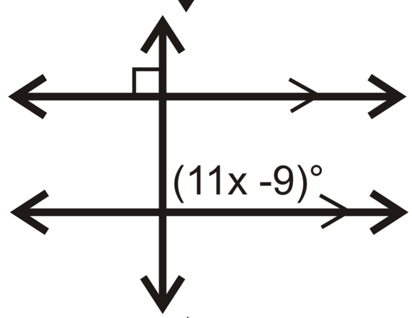

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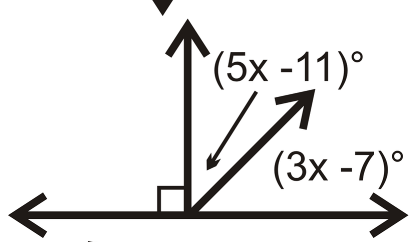

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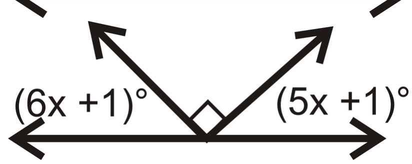
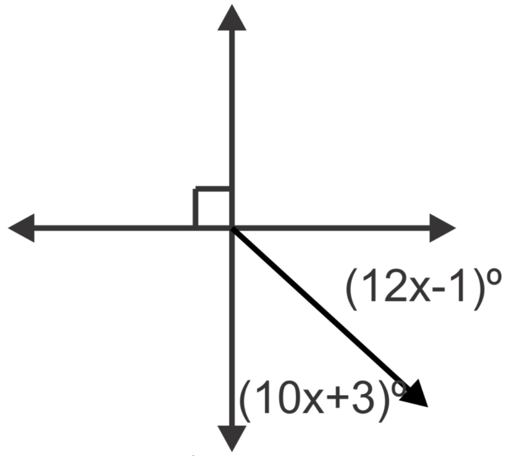
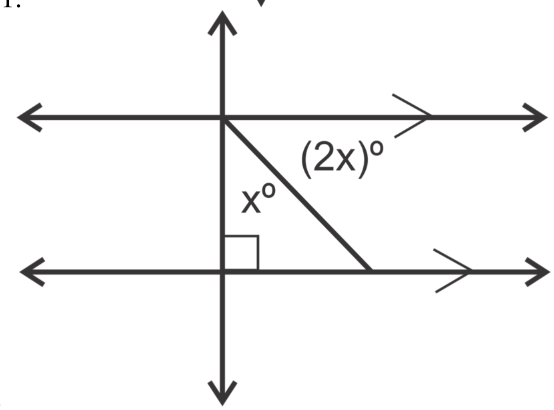


30.

31. 

32. 

Review Queue Answers

a. True; If four right angles are formed by two intersecting lines, then the lines are perpendicular.
b. False; 95° is not a right angle.
c. Undefined slope; this is a vertical line.
d. Zero slope; this would be a horizontal line.

3.5 Midpoints and Bisectors

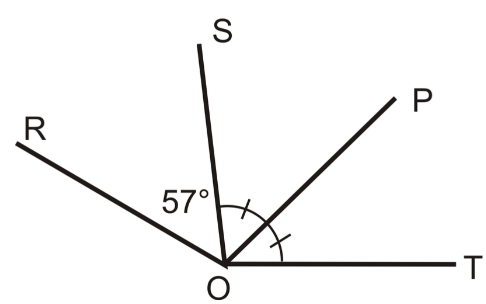
Learning Objectives

- Identify the midpoint of line segments.
- Identify the bisector of a line segment.
- Understand and the Angle Bisector Postulate.

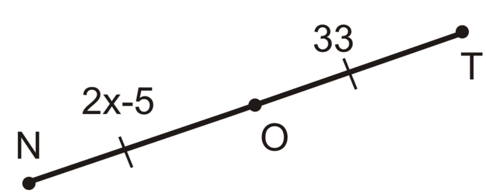
Review Queue

Answer the following questions.

a. $m\angle ROT = 165^\circ$, find $m\angle POT$



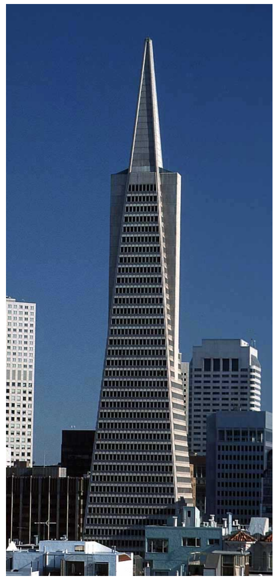
b. Find x.



c. Use the Angle Addition Postulate to write an equation for the angles in #1.

Know What? The building to the right is the TransamericaBuilding in San Francisco. This building was completed in 1972 and, at that time was one of the tallest buildings in the world. It is a pyramid with two "wings" on either side, to accommodate elevators. Because San Francisco has problems with earthquakes, there are regulations on how a building can be designed. In order to make this building as tall as it is and still abide by the building codes, the designer used this pyramid shape.

It is very important in designing buildings that the angles and parts of the building are equal. What components of this building look equal? Analyze angles, windows, and the sides of the building.



Congruence

You could argue that another word for ***equal*** is ***congruent***. However, the two differ slightly.

Congruent: When two geometric figures have the same shape and size.

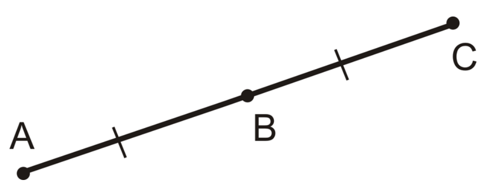
We label congruence with a $\cong$ sign. Notice the $\sim$ above the = sign. $\overline{AB} \cong \overline{BA}$ means that $\overline{AB}$ is congruent to $\overline{BA}$. If we know two segments or angles are congruent, then their measures are also equal. If two segments or angles have the same measure, then, they are also congruent.

TABLE 3.10:

Equal	***Congruent***
$=$	$\cong$
used with ***measurement***	used with ***figures***
$m\overline{AB} = AB = 5\ cm$	$\overline{AB} \cong \overline{BA}$
$m\angle ABC = 60^\circ$	$\angle ABC \cong \angle CBA$

Midpoints

Midpoint: A point on a line segment that divides it into two congruent segments.

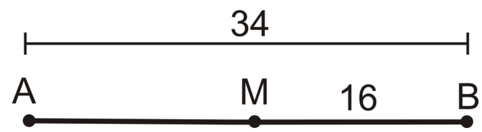


Because $AB = BC$, B is the midpoint of $\overline{AC}$.

Midpoint Postulate: Any line segment will have exactly one midpoint.

This might seem self-explanatory. However, be careful, this postulate is referring to the *midpoint,* not the lines that pass through the midpoint, which is infinitely many.

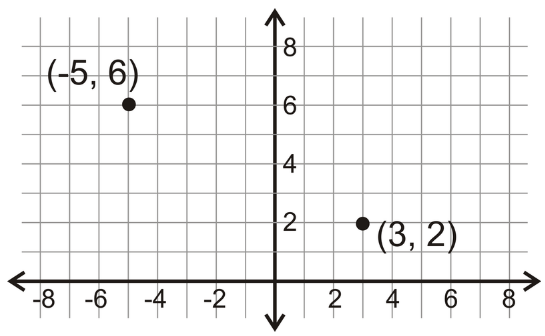
Example 1: Is M a midpoint of $\overline{AB}$?



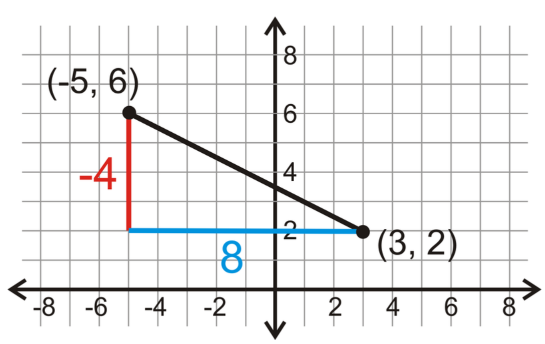
Solution: No, it is not because $MB = 16$ and $AM = 34 - 16 = 18$.

Midpoint Formula

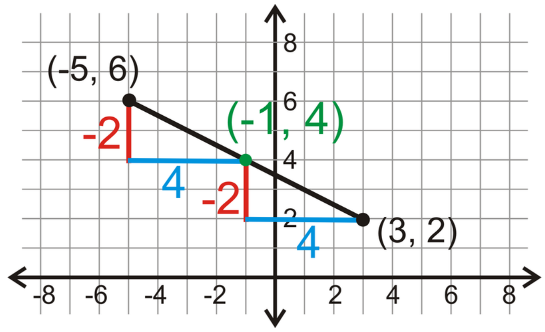
When points are plotted in the coordinate plane, you can use slope to find the midpoint between then. We will generate a formula here.



Here are two points, (-5, 6) and (3, 4). Draw a line between the two points and determine the vertical distance and the horizontal distance.



So, it follows that the midpoint is down and over half of each distance. The midpoint would then be down 2 (or -2) from (-5, 6) and over positively 4. If we do that we find that the midpoint is (-1, 4).



Let's create a formula from this. If the two endpoints are (-5, 6) and (3, 4), then the midpoint is (-1, 4). -1 is ***halfway*** between -5 and 3 and 4 is ***halfway*** between 6 and 2. Therefore, the formula for the midpoint is the average of the $x-$values and the average of the $y-$values.

Midpoint Formula: For two points, $(x_1, \ y_1)$ and $(x_2, \ y_2)$, the midpoint is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Example 2: Find the midpoint between (9, -2) and (-5, 14).

Solution: Plug the points into the formula.

$$\left(\frac{9+(-5)}{2}, \frac{-2+14}{2}\right) = \left(\frac{4}{2}, \frac{12}{2}\right) = (2,6)$$

Example 3: If $M(3,-1)$ is the midpoint of $\overline{AB}$ and $B(7,-6)$, find A.

Solution: Plug what you know into the midpoint formula.

$$\begin{aligned}\left(\frac{7+x_A}{2}, \frac{-6+y_A}{2}\right) &= (3,-1) \\ \frac{7+x_A}{2} = 3 \text{ and } \frac{-6+y_A}{2} &= -1 \qquad A \text{ is } (-1,4). \\ 7+x_A = 6 \text{ and } -6+y_A &= -2 \\ x_A = -1 \text{ and } y_A &= 4\end{aligned}$$

Another way to find the other endpoint is to find the difference between M and B and then duplicate it on the other side of M.

$x-$ *values*: $7-3=4$, so 4 on the other side of 3 is $3-4=-1$

$y-$ *values*: $-6-(-1)=-5$, so -5 on the other side of -1 is $-1-(-5)=4$

A is still (-1, 4). You may use either method.

Segment Bisectors

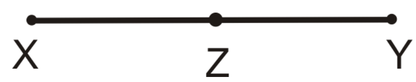
Segment Bisector: A line, segment, or ray that passes through a midpoint of another segment.

A bisector cuts a line segment into two congruent parts.

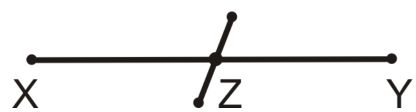
Example 4: Use a ruler to draw a bisector of the segment below.

Solution: The first step in identifying a bisector is finding the midpoint. Measure the line segment and it is 4 cm long. To find the midpoint, divide 4 by 2.

So, the midpoint will be 2 cm from either endpoint, or halfway between. Measure 2 cm from one endpoint and draw the midpoint.

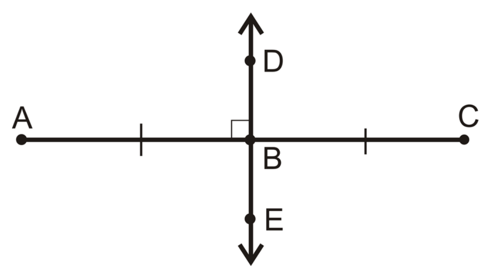


To finish, draw a line that passes through the midpoint. It doesn't matter how the line intersects $\overline{XY}$, as long as it passes through Z.



A specific type of segment bisector is called a perpendicular bisector.

Perpendicular Bisector: A line, ray or segment that passes through the midpoint of another segment and intersects the segment at a right angle.

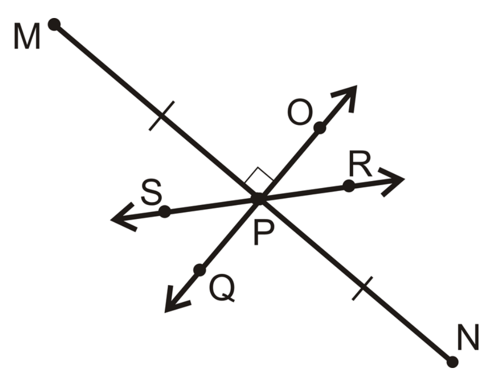


$\overleftrightarrow{DE}$ is the perpendicular bisector of $\overline{AC}$, so $\overline{AB} \cong \overline{BC}$ and $\overline{AC} \perp \overleftrightarrow{DE}$.

Perpendicular Bisector Postulate: For every line segment, there is one perpendicular bisector that passes through the midpoint.

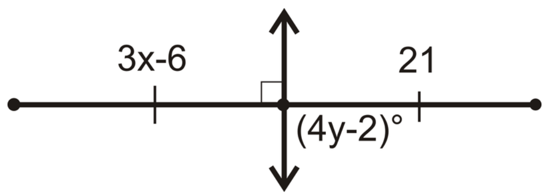
There are *infinitely many bisectors*, but only *one perpendicular bisector* for any segment.

Example 5: Which line is the perpendicular bisector of $\overline{MN}$?



Solution: The perpendicular bisector must bisect $\overline{MN}$ and be perpendicular to it. Only $\overleftrightarrow{OQ}$ satisfies both requirements. $\overleftrightarrow{SR}$ is just a bisector.

Example 6: ***Algebra Connection*** Find x and y.



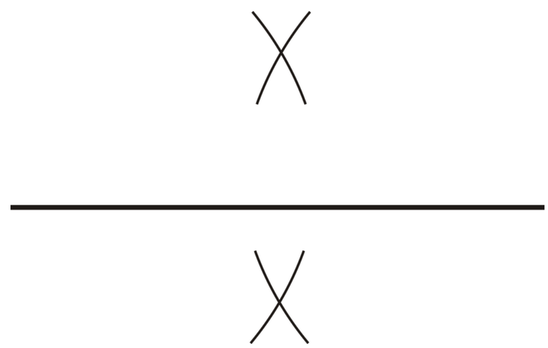
Solution: The line shown is the perpendicular bisector. So, $3x-6=21,\ 3x=27,\ x=9$.

And, $(4y-2)^\circ = 90^\circ, 4y^\circ = 92^\circ, y = 23^\circ$.

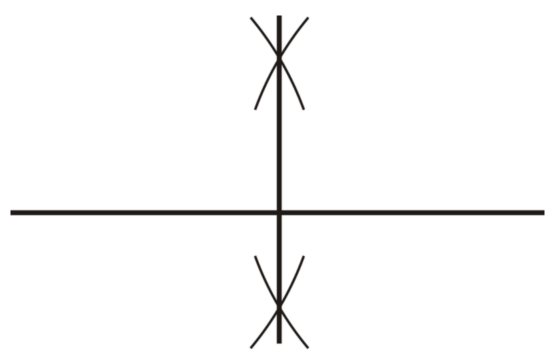
Investigation 1-3: Constructing a Perpendicular Bisector

a. Draw a line that is at least 6 cm long, about halfway down your page.

b. Place the pointer of the compass at an endpoint. Open the compass to be greater than half of the segment. Make arc marks above and below the segment. Repeat on the other endpoint. Make sure the arc marks intersect.



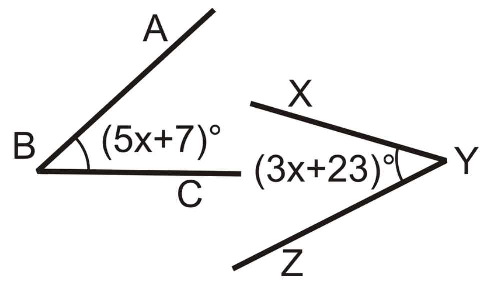
c. Use your straight edge to draw a line connecting the arc intersections.



This constructed line bisects the line you drew in #1 and intersects it at 90°. So, this construction also works to create a right angle. To see an animation of this investigation, go to http://www.mathsisfun.com/geometry/construct-linebisect.html .

Congruent Angles

Example 7: ***Algebra Connection*** What is the measure of each angle?



Solution: From the picture, we see that the angles are congruent, so the given measures are equal.

$$\begin{aligned}(5x+7)^\circ &= (3x+23)^\circ\\ 2x^\circ &= 16^\circ\\ x &= 8^\circ\end{aligned}$$

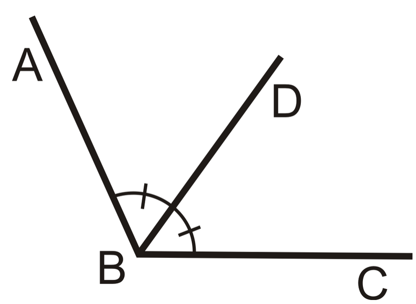
To find the measure of $\angle ABC$, plug in $x = 8^\circ$ to $(5x+7)^\circ$.

$$\begin{aligned}&(5(8)+7)^\circ\\ &(40+7)^\circ\\ &47^\circ\end{aligned}$$

Because $m\angle ABC = m\angle XYZ$, $m\angle XYZ = 47^\circ$ too.

Angle Bisectors

Angle Bisector: A ray that divides an angle into two congruent angles, each having a measure exactly half of the original angle.

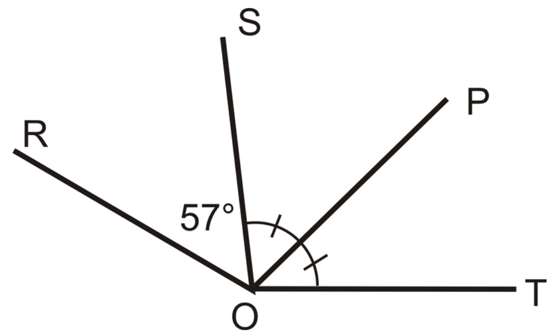


$\overline{BD}$ is the angle bisector of $\angle ABC$

$$\begin{aligned}\angle ABD &\cong \angle DBC\\ m\angle ABD &= \frac{1}{2} m\angle ABC\end{aligned}$$

Angle Bisector Postulate: Every angle has exactly one angle bisector.

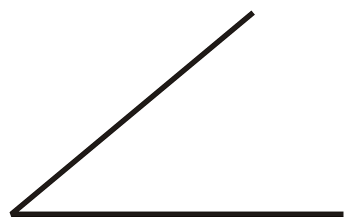
Example 8: Let's take a look at Review Queue #1 again. Is $\overline{OP}$ the angle bisector of $\angle SOT$? Recall, that $m\angle ROT = 165^\circ$, what is $m\angle SOP$ and $m\angle POT$?



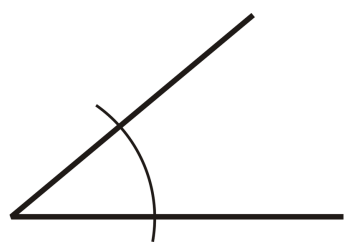
Solution: Yes, $\overline{OP}$ is the angle bisector of $\angle SOT$ according to the markings in the picture. If $m\angle ROT = 165^\circ$ and $m\angle ROS = 57^\circ$, then $m\angle SOT = 165^\circ - 57^\circ = 108^\circ$. The $m\angle SOP$ and $m\angle POT$ are each half of 108° or 54°.

Investigation 1-4: Constructing an Angle Bisector

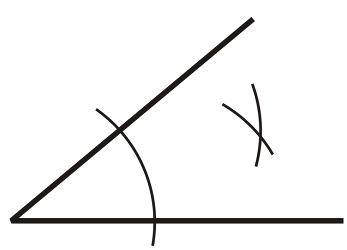
a. Draw an angle on your paper. Make sure one side is horizontal.



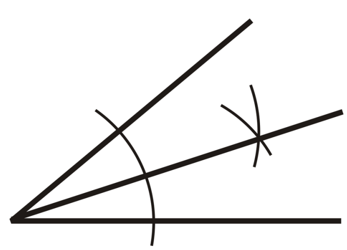
b. Place the pointer on the vertex. Draw an arc that intersects both sides.



c. Move the pointer to the arc intersection with the horizontal side. Make a second arc mark on the interior of the angle. Repeat on the other side. Make sure they intersect.



d. Connect the arc intersections from #3 with the vertex of the angle.

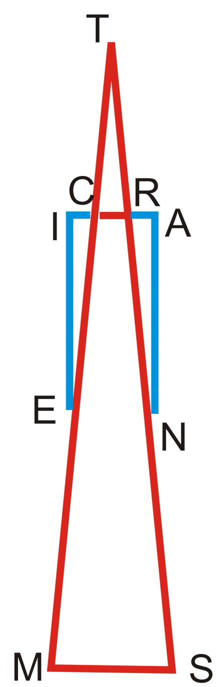


To see an animation of this construction, view http://www.mathsisfun.com/geometry/construct-anglebisect.html .

Know What? Revisited The image to the right is an outline of the Transamerica Building from earlier in the lesson. From this outline, we can see the following parts are congruent:

$$\begin{aligned} \overline{TR} &\cong \overline{TC} \\ \overline{RS} &\cong \overline{CM} \\ \overline{CI} &\cong \overline{RA} \\ \overline{AN} &\cong \overline{IE} \\ \overline{TS} &\cong \overline{TM} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \angle TCR &\cong \angle TRC \\ \angle CIE &\cong \angle RAN \\ \angle TMS &\cong \angle TSM \\ \angle IEC &\cong \angle ANR \\ \angle TCI &\cong \angle TRA \end{aligned}$$

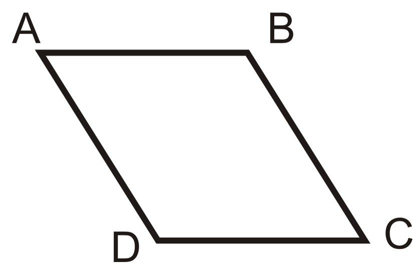
As well at these components, there are certain windows that are congruent and all four triangular sides of the building are congruent to each other.



Review Questions

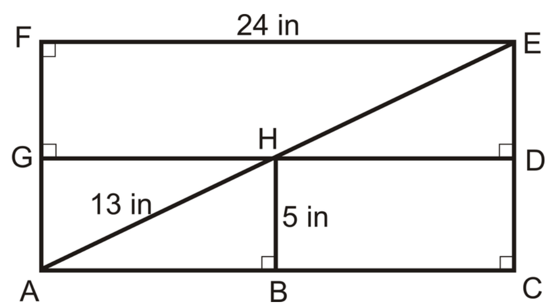
1. Copy the figure below and label it with the following information:

$$\begin{aligned} \angle A &\cong \angle C \\ \angle B &\cong \angle D \\ \overline{AB} &\cong \overline{CD} \\ \overline{AD} &\cong \overline{BC} \end{aligned}$$



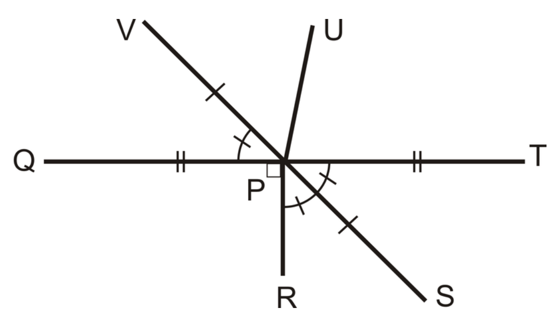
For 2-9, find the lengths, given: H is the midpoint of $\overline{AE}$ and $\overline{DG}$, B is the midpoint of $\overline{AC}$, $\overline{GD}$ is the perpendicular bisector of $\overline{FA}$ and $\overline{EC}$, $\overline{AC} \cong \overline{FE}$, and $\overline{FA} \cong \overline{EC}$.

2. AB
3. GA
4. ED
5. HE
6. $m\angle HDC$
7. FA
8. GD
9. $m\angle FED$



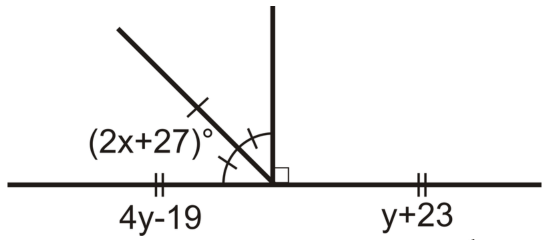
10. How many copies of triangle AHB can fit inside rectangle $FECA$ without overlapping?

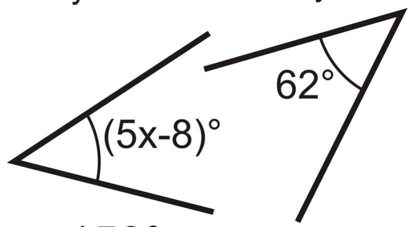
For 11-18, use the following picture to answer the questions.

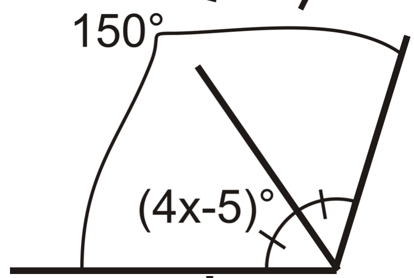


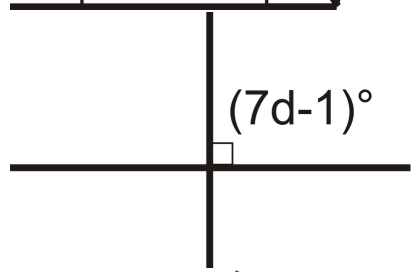
11. What is the angle bisector of $\angle TPR$?
12. P is the midpoint of what two segments?
13. What is $m\angle QPR$?
14. What is $m\angle TPS$?
15. How does $\overline{VS}$ relate to $\overline{QT}$?
16. How does $\overline{QT}$ relate to $\overline{VS}$?
17. Is $\overline{PU}$ a bisector? If so, of what?
18. What is $m\angle QPV$?

Algebra Connection For 19-24, use algebra to determine the value of variable(s) in each problem.

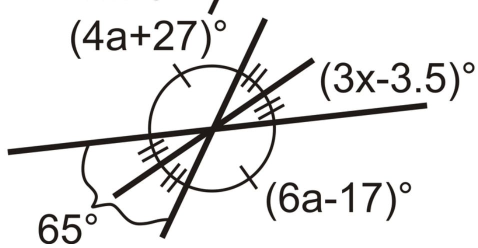
19. 

20. 

21. 

22. 

23. 

24. 

25. ***Construction*** Using your protractor, draw an angle that is 110°. Then, use your compass to construct the angle bisector. What is the measure of each angle?
26. ***Construction*** Using your protractor, draw an angle that is 75°. Then, use your compass to construct the angle bisector. What is the measure of each angle?
27. ***Construction*** Using your ruler, draw a line segment that is 7 cm long. Then use your compass to construct the perpendicular bisector, What is the measure of each segment?
28. ***Construction*** Using your ruler, draw a line segment that is 4 in long. Then use your compass to construct the perpendicular bisector, What is the measure of each segment?
29. ***Construction*** Draw a straight angle (180°). Then, use your compass to construct the angle bisector. What kind of angle did you just construct?

For questions 30-33, find the midpoint between each pair of points.

30. (-2, -3) and (8, -7)
31. (9, -1) and (-6, -11)
32. (-4, 10) and (14, 0)
33. (0, -5) and (-9, 9)

Given the midpoint (M) and either endpoint of $\overline{AB}$, find the other endpoint.

34. $A(-1,2)$ and $M(3,6)$
35. $B(-10,-7)$ and $M(-2,1)$
36. ***Error Analysis*** Erica is looking at a geometric figure and trying to determine which parts are congruent. She wrote $\overline{AB} = \overline{CD}$. Is this correct? Why or why not?
37. ***Challenge*** Use the Midpoint Formula to solve for the $x-$value of the midpoint and the $y-$value of the midpoint. Then, use this formula to solve #34. Do you get the same answer?
38. ***Construction Challenge*** Use construction tools and the constructions you have learned in this section to construct a 45° angle.
39. ***Construction Challenge*** Use construction tools and the constructions you have learned in this section to construct two 2 in segments that bisect each other. Now connect all four endpoints with segments. What figure have you constructed?
40. Describe an example of how the concept of midpoint (or the midpoint formula) could be used in the real world.

Review Queue Answers

a. See Example 6
b. $2x-5=33$
   $2x=38$
   $x=19$
c. $m\angle ROT = m\angle ROS + m\angle SOP + m\angle POT$

3.6 Finding the Slope and Equation of a Line

Objective

To review how to find the slope and equation of a line.

Review Queue

1. Plot the following points on the same graph.

a) (4, -2)

b) (-2, -7)

c) (6, 1)

d) (0, 8)

Solve the following equations for the indicated variable.

2. $3x-4y=12; x$

3. $3x-4y=12; y$

4. $2b+5c=-10; b$

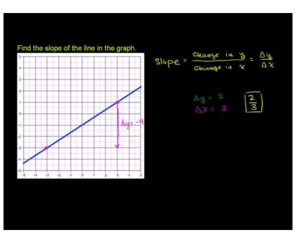
5. $2b+5c=-10; c$

Finding Slope

Objective

To find the slope of a line and between two points.

Watch This



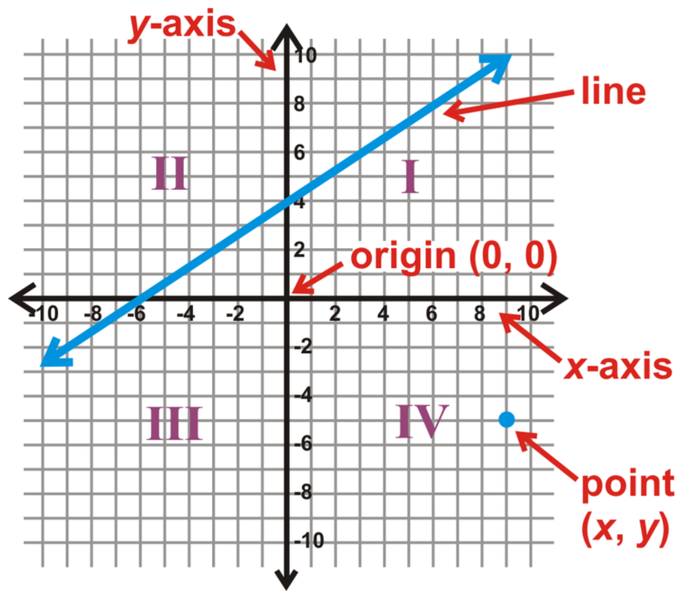
MEDIA

Click image to the left for more content.

Khan Academy: Slope of a line

Guidance

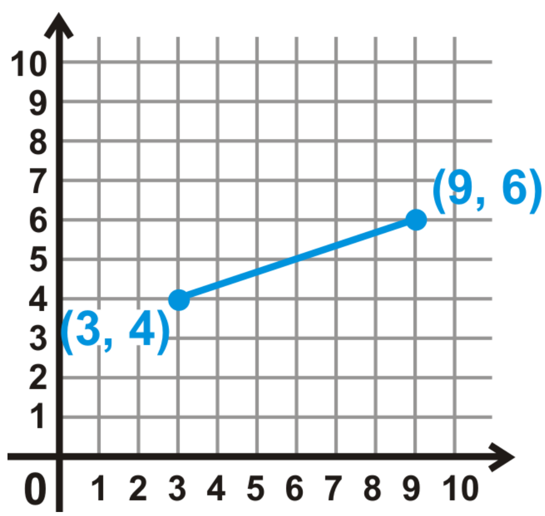
The slope of a line determines how steep or flat it is. When we place a line in the coordinate plane, we can measure the slope, or steepness, of a line. Recall the parts of the coordinate plane, also called an $x-y$ plane and the Cartesian plane, after the mathematician Descartes.



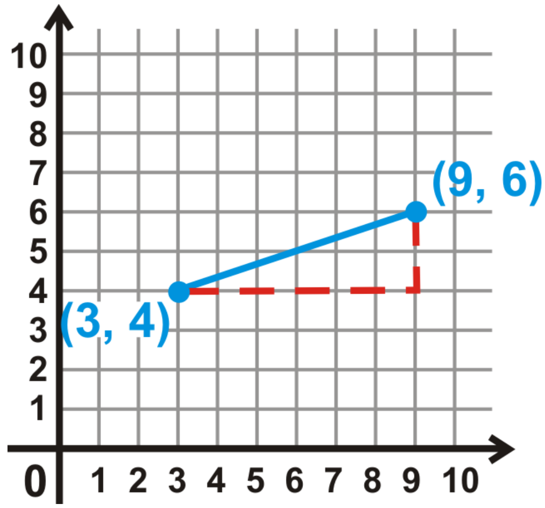
To plot a point, order matters. First, every point is written (x, y), where x is the movement in the $x-$direction and y is the movement in the $y-$direction. If x is negative, the point will be in the 2^{nd} or 3^{rd} quadrants. If y is negative, the point will be in the 3^{rd} or 4^{th} quadrants. The quadrants are always labeled in a counter-clockwise direction and using Roman numerals.

The point in the 4^{th} quadrant would be (9, -5).

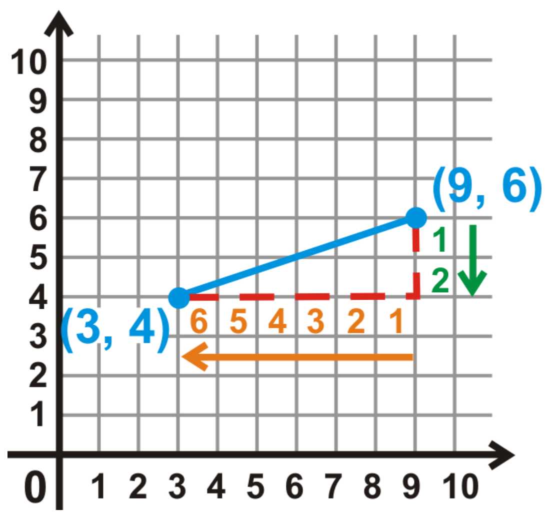
To find the slope of a line or between two points, first, we start with right triangles. Let's take the two points (9, 6) and (3, 4). Plotting them on an $x-y$ plane, we have:



To turn this segment into a right triangle, draw a vertical line down from the higher point, and a horizontal line from the lower point, towards the vertical line. Where the two lines intersect is the third vertex of the slope triangle.



Now, count the vertical and horizontal units along the horizontal and vertical sides (**red** dotted lines).

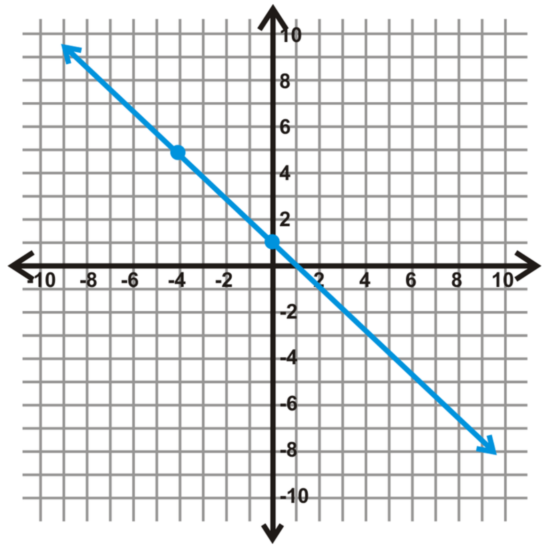


The slope is a fraction with the vertical distance over the horizontal distance, also called the "rise over run." Because the vertical distance goes down, we say that it is -2. The horizontal distance moves towards the negative direction (the left), so we would say that it is -6. So, for slope between these two points, the slope would be $\frac{-2}{-6}$ or $\frac{1}{3}$.

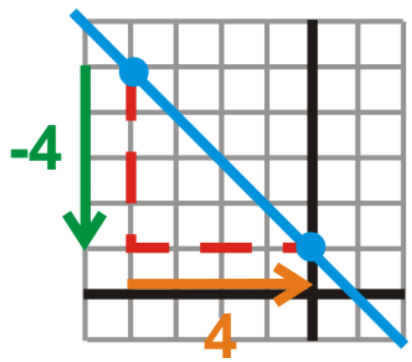
Note: You can also draw the right triangle above the line segment.

Example A

Use a slope triangle to find the slope of the line below.



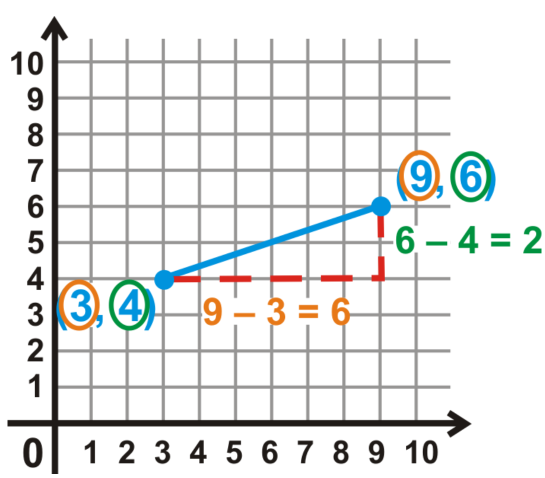
Solution: Notice the two points that are drawn on the line. These are given to help you find the slope. Draw a triangle between these points and find the slope.



From the slope triangle above, we see that the slope is $\frac{-4}{4} = -1$.

Whenever a slope reduces to a whole number, the "run" will always be positive 1. Also, notice that this line points in the opposite direction as the line segment above. We say this line has a *negative* slope because the slope is a negative number and points from the 2^{nd} to 4^{th} quadrants. A line with positive slope will point in the opposite direction and point between the 1^{st} and 3^{rd} quadrants.

If we go back to our previous example with points (9, 6) and (3, 4), we can find the vertical distance and horizontal distance another way.



From the picture, we see that the vertical distance is the same as the difference between the $y-$values and the horizontal distance is the difference between the $x-$values. Therefore, the slope is $\frac{6-4}{9-3}$. We can extend this idea to any two points, (x_1, y_1) and (x_2, y_2).

Slope Formula: For two points (x_1, y_1) and (x_2, y_2), the slope between them is $\frac{y_2-y_1}{x_2-x_1}$. The symbol for slope is m.

It does not matter which point you choose as (x_1, y_1) or (x_2, y_2).

Example B

Find the slope between (-4, 1) and (6, -5).

Solution: Use the Slope Formula above. Set $(x_1, y_1) = (-4, 1)$ and $(x_2, y_2) = (6, -5)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-(-4)}{-5-1} = \frac{10}{-6} = -\frac{5}{3}$$

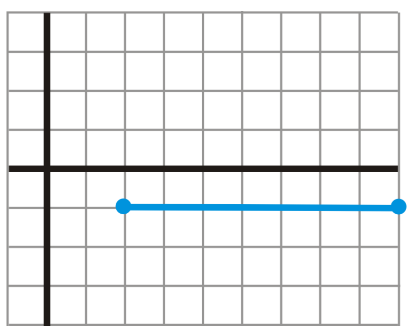
Example C

Find the slope between (9, -1) and (2, -1).

Solution: Use the Slope Formula. Set $(x_1, y_1) = (9, -1)$ and $(x_2, y_2) = (2, -1)$.

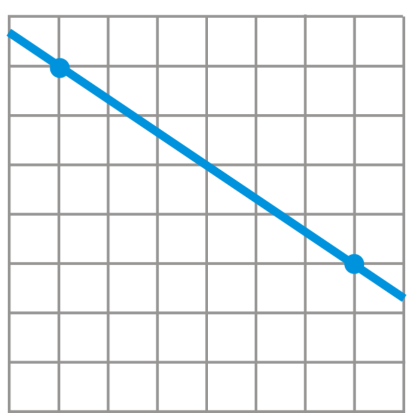
$$m = \frac{-1-(-1)}{2-9} = \frac{0}{-7} = 0$$

Here, we have zero slope. Plotting these two points we have a horizontal line. This is because the $y-$values are the same. Anytime the $y-$values are the same we will have a horizontal line and the slope will be zero.



Guided Practice

1. Use a slope triangle to find the slope of the line below.



2. Find the slope between (2, 7) and (-3, -3).

3. Find the slope between (-4, 5) and (-4, -1).

Answers

1. Counting the squares, the vertical distance is down 6, or -6, and the horizontal distance is to the right 8, or +8. The slope is then $\frac{-6}{8}$ or $-\frac{2}{3}$.

2. Use the Slope Formula. Set $(x_1, y_1) = (2, 7)$ and $(x_2, y_2) = (-3, -3)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-7}{-3-2} = \frac{-10}{-5} = 2$$

3. Again, use the Slope Formula. Set $(x_1, y_1) = (-4, 5)$ and $(x_2, y_2) = (-4, -1)$.

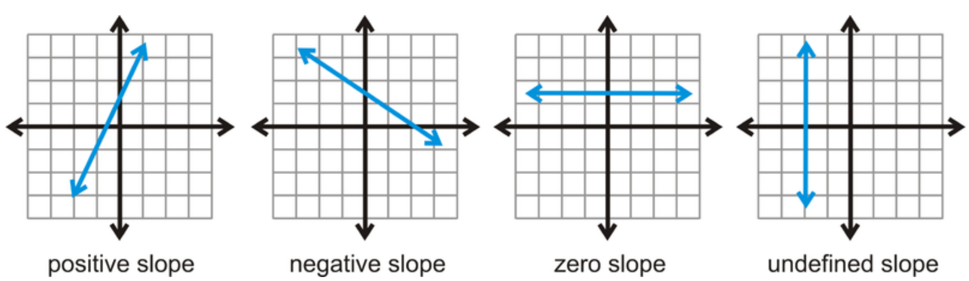
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-5}{-4-(-4)} = \frac{-6}{0}$$

You cannot divide by zero. Therefore, this slope is undefined. If you were to plot these points, you would find they form a vertical line. *All vertical lines have an undefined slope.*

Important Note: Always reduce your slope fractions. Also, if the numerator or denominator of a slope is negative, then the slope is negative. If they are both negative, then we have a negative number divided by a negative number, which is positive, thus a positive slope.

Vocabulary

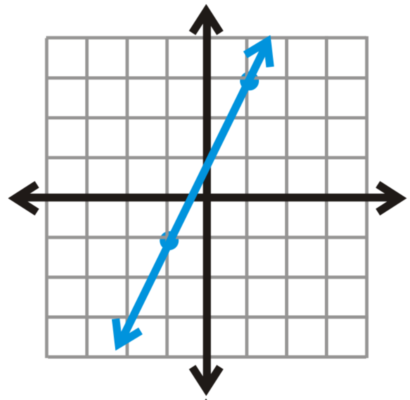
Slope
The steepness of a line. A line can have positive, negative, zero (horizontal), or undefined (vertical) slope. Slope can also be called "rise over run" or "the change in the $y-$values over the change in the $x-$values." The symbol for slope is m.

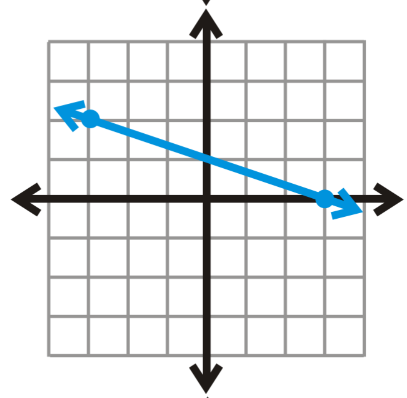


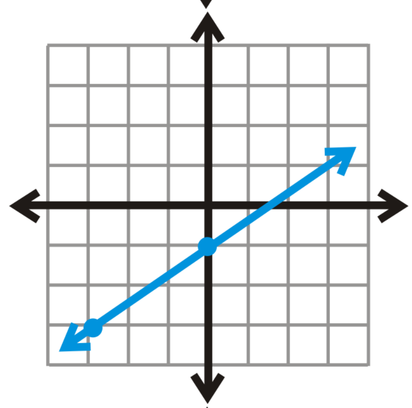
Slope Formula
For two points (x_1, y_1) and (x_2, y_2), the slope between them is $\frac{y_2 - y_1}{x_2 - x_1}$.

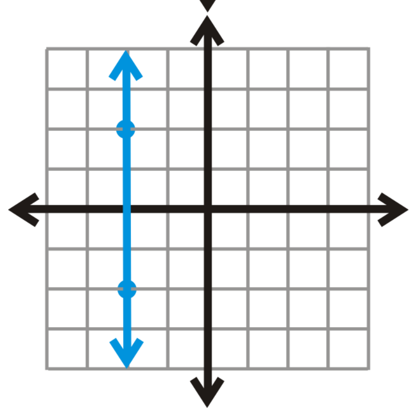
Problem Set

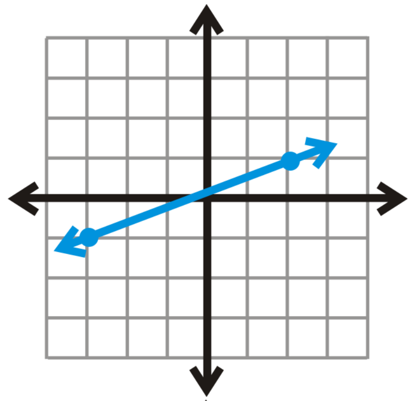
Find the slope of each line by using slope triangles.

1. 

2. 

3. 

4. 



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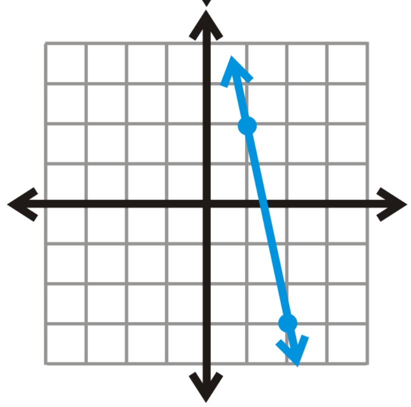


6.

Find the slope between each pair of points using the Slope Formula.

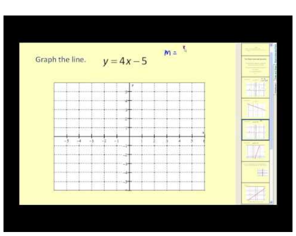
7. (-5, 6) and (-3, 0)
8. (1, -1) and (6, -1)
9. (3, 2) and (-9, -2)
10. (8, -4) and (8, 1)
11. (10, 2) and (4, 3)
12. (-3, -7) and (-6, -3)
13. (4, -5) and (0, -13)
14. (4, -15) and (-6, -11)
15. (12, 7) and (10, -1)
16. **Challenge** The slope between two points (a,b) and (1, -2) is $\frac{1}{2}$. Find a and b.

Finding the Equation of a Line in Slope-Intercept Form

Objective

To find the equation of a line (the slope and $y-$intercept) in slope-intercept form.

Watch This



MEDIA

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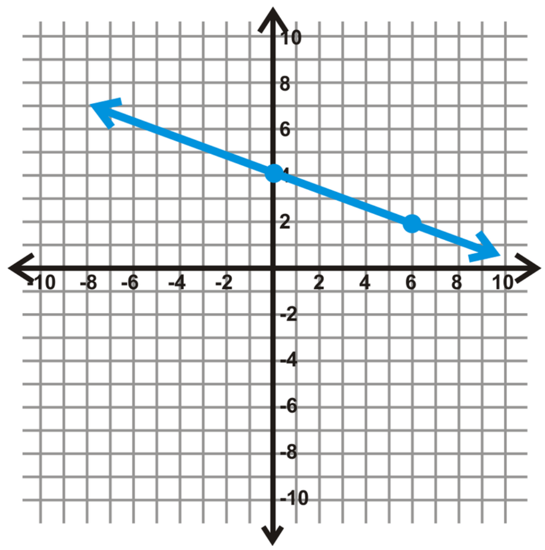
James Sousa: Slope Intercept Form of a Line

Guidance

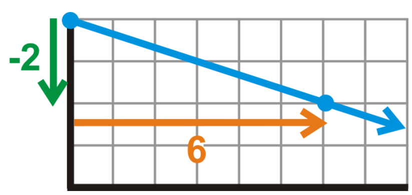
In the previous concept, we found the slope between two points. We will now find the entire equation of a line. Recall from Algebra I that the equation of a line in slope-intercept form is $y = mx + b$, where m is the slope and b is the $y-$intercept. You can find the slope either by using slope triangles or the Slope Formula. To find the $y-$intercept, or b, you can either locate where the line crosses the $y-$axis (if given the graph) or by using algebra.

Example A

Find the equation of the line below.



Solution: Analyze the line. We are given two points on the line, one of which is the $y-$intercept. From the graph, it looks like the line passes through the $y-$axis at (0, 4), making $b = 4$. Now, we need to find the slope. You can use slope triangles or the Slope Formula. Using slope triangles, we have:



From this, we see that the slope is $-\frac{2}{6}$ or $-\frac{1}{3}$.

Plugging our found information into the slope-intercept equation, the equation of this line is $y = -\frac{1}{3}x + 4$.

Alternate Method: If we had used the Slope Formula, we would use (0, 4) and (6, 2), which are the values of the given points.

$$m = \frac{2-4}{6-0} = \frac{-2}{6} = -\frac{1}{3}$$

Example B

The slope of a line is -4 and the $y-$intercept is (0, 3). What is the equation of the line?

Solution: This problem explicitly tells us the slope and $y-$intercept. The slope is -4, meaning $m = -4$. The $y-$intercept is (0, 3), meaning $b = 3$. Therefore, the equation of the line is $y = -4x + 3$.

Example C

The slope of a line is $\frac{1}{2}$ and it passes through the point (4, -7). What is the equation of the line?

Solution: In this problem, we are given m and a point on the line. The point, (4, -7) can be substituted in for x and y in the equation. We need to solve for the $y-$intercept, or b. Plug in what you know to the slope-intercept equation.

$$\begin{aligned}y &= mx+b\\ -7 &= \frac{1}{2}(4)+b\\ -7 &= 2+b\\ -9 &= b\end{aligned}$$

From this, the equation of the line is $y=\frac{1}{2}x-9$.

We can test if a point is on a line or not by plugging it into the equation. If the equation holds true, the point is on the line. If not, then the point is not on the line.

Example D

Find the equation of the line that passes through (12, 7) and (10, -1).

Solution: In this example, we are not given the slope or the $y-$intercept. First, we need to find the slope using the Slope Formula.

$$m=\frac{-1-7}{10-12}=\frac{-8}{-2}=4$$

Now, plug in one of the points for x and y. It does not matter which point you choose because they are both on the line.

$$\begin{aligned}7 &= 4(12)+b\\ 7 &= 48+b\\ -41 &= b\end{aligned}$$

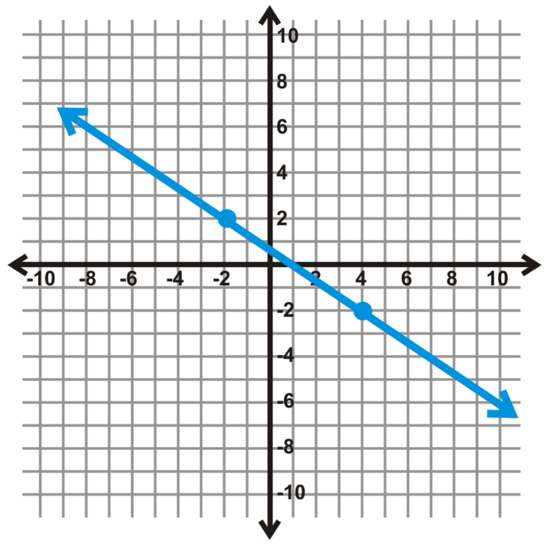
The equation of the line is $y=4x-41$.

Guided Practice

1. What is the equation of the line where the slope is 1 and passes through (5, 3)?

2. Find the equation of the line that passes through (9, -4) and (-1, -8).

3. Find the equation of the line below.



Answers

1. We are told that $m = 1, x = 5$, and $y = 3$. Plug this into the slope-intercept equation and solve for b.

$$\begin{aligned} 3 &= 1(5) + b \\ 3 &= 5 + b \\ -2 &= b \end{aligned}$$

The equation of the line is $y = x - 2$

2. First, find the slope.

$$m = \frac{-8-(-4)}{-1-9} = \frac{-4}{-10} = \frac{2}{5}$$

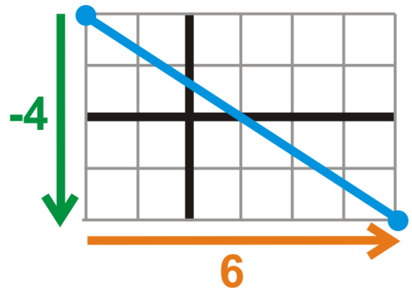
Now, find the $y-$intercept. We will use the second point. Remember, it does not matter which point you use.

$$\begin{aligned} -8 &= \frac{2}{5}(-1) + b \\ -8 &= -\frac{2}{5} + b \\ -7\frac{3}{5} &= b \end{aligned}$$

The equation of the line is $y = \frac{2}{5}x - 7\frac{3}{5}$ or $y = \frac{2}{5}x - \frac{38}{5}$.

When your $y-$intercept is a fraction, make sure it is reduced. Double-check with your teacher on how s/he wants you to leave your answer.

3. We can find the slope one of two ways: using slope triangles or by using the Slope Formula. We are given (by the drawn points in the picture) that (-2, 2) and (4, -2) are on the line. Drawing a slope triangle, we have:



We have that the slope is $-\frac{4}{6}$ or $-\frac{2}{3}$. To find the $y-$intercept, it looks like it is somewhere between 0 and 1. Take one of the points and plug in what you know to the slope-intercept equation.

$$2 = -\frac{2}{3}(-2) + b$$
$$2 = \frac{4}{3} + b$$
$$\frac{2}{3} = b$$

The equation of the line is $y = -\frac{2}{3}x + \frac{2}{3}$.

Vocabulary

Slope-Intercept Form
The equation of a line in the form $y = mx + b$, where m is the slope and b is the $y-$intercept.

$y-$**intercept**
The point where a line crosses the $y-$axis. This point will always have the form $(0, y)$.

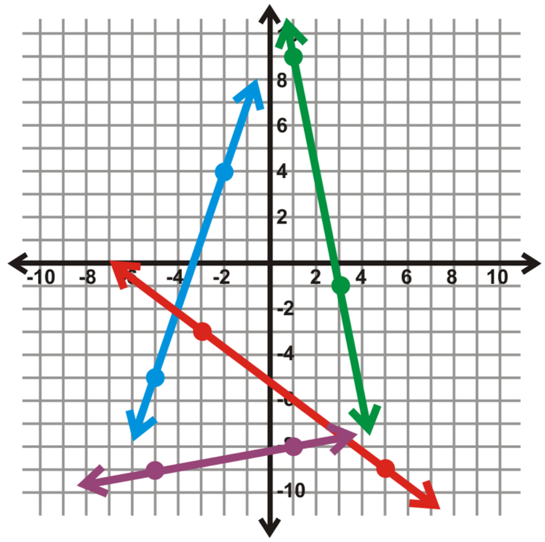
$x-$**intercept**
The point where a line crosses the $x-$axis. This point will always have the form $(x, 0)$.

Problem Set

Find the equation of each line with the given information below.

1. slope = 2, $y-$intercept = (0, 3)
2. $m = -\frac{1}{4}$, $b = 2.6$
3. slope = -1, $y-$intercept = (0, 2)
4. $x-$intercept = (-2, 0), $y-$intercept = (0, -5)
5. slope $= \frac{2}{3}$ and passes through (6, -4)
6. slope $= -\frac{3}{4}$ and passes through (-2, 5)
7. slope = -3 and passes through (-1, -7)
8. slope = 1 and passes through (2, 4)
9. passes through (-5, 4) and (1, 1)
10. passes through (5, -1) and (-10, -10)
11. passes through (-3, 8) and (6, 5)
12. passes through (-4, -21) and (2, 9)

For problems 13-16, find the equation of the lines in the graph below.



13. Green Line
14. Blue Line
15. Red Line
16. Purple Line
17. Find the equation of the line with zero slope and passes through (8, -3).
18. Find the equation of the line with zero slope and passes through the point (-4, 5).
19. Find the equation of the line with zero slope and passes through the point (a,b).
20. **Challenge** Find the equation of the line with an *undefined* slope that passes through (a,b).

3.7 Standard Form of a Line

Objective

To familiarize students with the standard form of a line, as well as finding the equations of lines that are parallel or perpendicular to a given line.

Review Queue

1. Solve $4x - y = 6$ for y.
2. Solve $4x - 8y = 12$ for y.
3. What are the slope and $y-$intercept of $y = -\frac{2}{3}x + 5$?
4. Define *parallel* and *perpendicular* in your own words.

Standard Form

Objective

To manipulate and use the standard form of a line.

Guidance

Slope-intercept form is one way to write the equation of a line. Another way is called standard form. Standard form looks like $Ax + By = C$, where A, B, and C are all integers. In the Review Queue above, the equations from problems 1 and 2 are in standard form. Once they are solved for y, they will be in slope-intercept form.

Example A

Find the equation of a line, in standard form, where the slope is $\frac{3}{4}$ and passes through (4, -1).

Solution: To find the equation in standard form, you need to determine what A, B, and C are. Let's start this example by finding the equation in slope-intercept form.

$$\begin{aligned}
-1 &= \frac{3}{4}(4) + b \\
-1 &= 3 + b \\
-4 &= b
\end{aligned}$$

In slope-intercept form, the equation is $y = \frac{3}{4}x - 4$.

To change this to standard form we need to subtract the $x-$term from both sides of the equation.

$$-\frac{3}{4}x + y = -4$$

However, we are not done. In the definition, A, B, and C are all integers. At the moment, A is a fraction. To undo the fraction, we must multiply all the terms by the denominator, 4. We also will multiply by a negative so that the $x-$coefficient will be positive.

$$-4\left(-\frac{3}{4}x+y=-4\right)$$
$$3x-4y=16$$

Example B

The equation of a line is $5x-2y=12$. What are the slope and $y-$intercept?

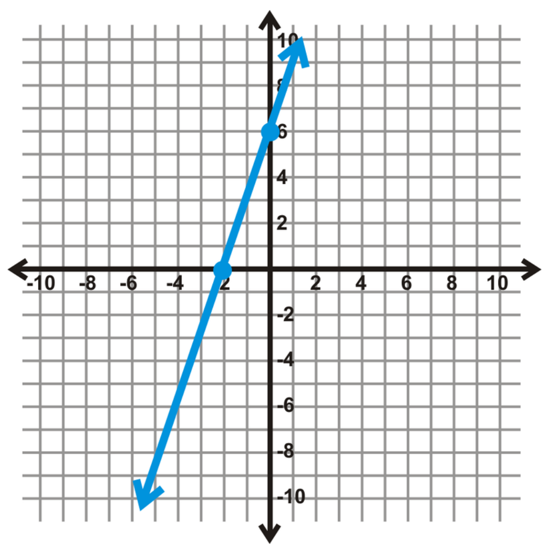
Solution: To find the slope and $y-$intercept of a line in standard form, we need to switch it to slope-intercept form. This means, we need to solve the equation for y.

$$\begin{aligned}5x-2y&=12\\-2y&=-5x+12\\y&=\frac{5}{2}x-6\end{aligned}$$

From this, the slope is $\frac{5}{2}$ and the $y-$intercept is (0, -6).

Example C

Find the equation of the line below, in standard form.



Solution: Here, we are given the intercepts. The slope triangle is drawn by the axes, $\frac{-6}{-2}=3$. And, the $y-$intercept is (0, 6). The equation of the line, in slope-intercept form, is $y=3x+6$. To change the equation to standard form, subtract the $x-$term to move it over to the other side.

$$-3x+y=6 \ or \ 3x-y=-6$$

Example D

The equation of a line is $6x-5y=45$. What are the intercepts?

Solution: For the $x-$intercept, the $y-$value is zero. Plug in zero for y and solve for x.

$$\begin{aligned} 6x - 5y &= 45 \\ 6x - 5(0) &= 45 \\ 6x &= 45 \\ x &= \frac{45}{6} \ or \ \frac{15}{2} \end{aligned}$$

The $x-$intercept is $\left(\frac{15}{2}, 0\right)$.

For the $y-$intercept, the $x-$value is zero. Plug in zero for x and solve for y.

$$\begin{aligned} 6x - 5y &= 45 \\ 6(0) - 5y &= 45 \\ 5y &= 45 \\ y &= 9 \end{aligned}$$

The $y-$intercept is (0, 9).

Guided Practice

1. Find the equation of the line, in standard form that passes through (8, -1) and (-4, 2).

2. Change $2x+3y=9$ to slope-intercept form.

3. What are the intercepts of $3x-4y=-24$?

Answers

1. Like with Example A, we need to first find the equation of this line in $y-$intercept form and then change it to standard form. First, find the slope.

$$\frac{2-(-1)}{-4-8} = \frac{3}{-12} = -\frac{1}{4}$$

Find the $y-$intercept using slope-intercept form.

$$\begin{aligned} 2 &= -\frac{1}{4}(-4) + b \\ 2 &= 1 + b \\ 1 &= b \end{aligned}$$

The equation of the line is $y = -\frac{1}{4}x + 1$.

To change this equation into standard form, add the $x-$term to both sides and multiply by 4 to get rid of the fraction.

$$\begin{aligned} \frac{1}{4}x + y &= 1 \\ 4\left(\frac{1}{4}x + y = 1\right) \\ x + 4y &= 1 \end{aligned}$$

2. To change $2x+3y=9$ into slope-intercept form, solve for y.

$$\begin{aligned}2x+3y&=9\\3y&=-2x+9\\y&=-\frac{2}{3}x+3\end{aligned}$$

3. Copy Example D to find the intercepts of $3x-4y=-24$. First, plug in zero for y and solve for x.

$$\begin{aligned}3x-4(0)&=-24\\3x&=-24\\x&=-8\end{aligned}$$

$x-$intercept is (-8, 0)

Now, start over and plug in zero for x and solve for y.

$$\begin{aligned}3(0)-4y&=-24\\-4y&=-24\\y&=6\end{aligned}$$

$y-$intercept is (6, 0)

Vocabulary

Standard Form (of a line)
When a line is in the form $Ax+By=C$ and A,B, and C are integers.

Problem Set

Change the following equations into standard form.

1. $y=-\frac{2}{3}x+4$
2. $y=x-5$
3. $y=\frac{1}{5}x-1$

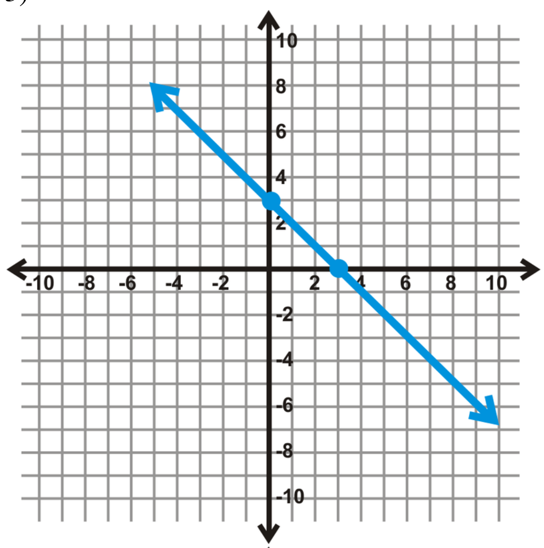
Change the following equations into slope-intercept form.

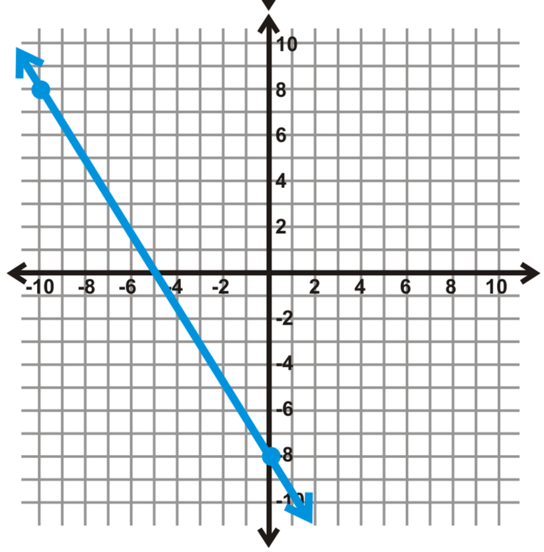
4. $4x+5y=20$
5. $x-2y=9$
6. $2x-3y=15$

Find the x and $y-$intercepts of the following equations.

7. $3x+4y=12$
8. $6x-y=8$
9. $3x+8y=-16$

Find the equation of the lines below, in standard form.

10. slope = 2 and passes through (3, -5)
11. slope $= -\frac{1}{2}$ and passes through (6, -3).
12. passes through (5, -7) and (-1, 2)
13. passes through (-5, -5) and (5, -3)
14. 

15. 

16. Change $Ax+By=C$ into slope-intercept form.
17. From #16, what are the slope and $y-$intercept equal to (in terms of A, B, and/or C)?
18. Using #16 and #17, find one possible combination of A, B, and C for $y = \frac{1}{2}x - 4$. Write your answer in standard form.
19. The measure of a road's slope is called the *grade*. The grade of a road is measured in a percentage, for how many vertical feet the road rises or declines over 100 feet. For example, a road with a grade incline of 5% means that for every 100 horizontal feet the road rises 5 vertical feet. What is the slope of a road with a grade decline of 8%?
20. The population of a small town in northern California gradually increases by about 50 people a year. In 2010, the population was 8500 people. Write an equation for the population of this city and find its estimated population in 2017.

Finding the Equation of Parallel Lines

Objective

To find the equation of a line that is parallel to a given line.

Guidance

When two lines are parallel, they have the same slope and never intersect. So, if a given line has a slope of -2, then any line that is parallel to that line will also have a slope of -2, but it will have a different $y-$intercept.

Example A

Find the equation of the line that is parallel to $y = \frac{2}{3}x - 5$ and passes through (-12, 1).

Solution: We know that the slopes will be the same; however we need to find the $y-$intercept for this new line. Use the point you were given, (-12, 1) and plug it in for x and y to solve for b.

$$\begin{aligned} y &= \frac{2}{3}x + b \\ 1 &= \frac{2}{3}(-12) + b \\ 1 &= -8 + b \\ 9 &= b \end{aligned}$$

The equation of the parallel line is $y = \frac{2}{3}x + 9$.

Example B

Write the equation of the line that passes through (4, -7) and is parallel to $y = -2$.

Solution: The line $y = -2$ does not have an $x-$term, meaning it has no slope. This is a horizontal line. Therefore, to find the horizontal line that passes through (4, -7), we only need the $y-$coordinate. The line would be $y = -7$.

The same would be true for vertical lines, but all vertical line equations are in the form $x = a$. The $x-$coordinate of a given point would be what is needed to determine the equation of the parallel vertical line.

Example C

Write the equation of the line that passes through (6, -10) and is parallel to the line that passes through (4, -6) and (3, -4).

Solution: First, we need to find the slope of the line that our line will be parallel to. Use the points (4, -6) and (3, -4) to find the slope.

$$m = \frac{-4 - (-6)}{3 - 4} = \frac{2}{-1} = -2$$

This is the slope of our given line as well as the parallel line. Use the point (6, -10) to find the $y-$intercept of the line that we are trying to find the equation for.

$$\begin{aligned} -10 &= -2(6) + b \\ -10 &= -12 + b \\ 2 &= b \end{aligned}$$

The equation of the line is $y = -2x + 2$.

Guided Practice

1. Find the equation of the line that is parallel to $x - 2y = 8$ and passes through (4, -3).

2. Find the equation of the line that is parallel to $x = 9$ and passes through (-1, 3).

3. Find the equation of the line that passes through (-5, 2) and is parallel to the line that passes through (6, -1) and (1, 3).

Answers

1. First, we need to change this line from standard form to slope-intercept form.

$$\begin{aligned} x - 2y &= 8 \\ -2y &= -x + 8 \quad \text{Now, we know the slope is } \frac{1}{2}\text{. Let's find the new } y- \text{ intercept.} \\ y &= \frac{1}{2}x - 4 \end{aligned}$$

$$\begin{aligned} -3 &= \frac{1}{2}(4) + b \\ -3 &= 2 + b \\ -5 &= b \end{aligned}$$

The equation of the parallel line is $y = \frac{1}{2}x - 5$ or $x - 2y = 10$.

2. $x = 9$ is a vertical line that passes through the $x-$axis at 9. Therefore, we only need to $x-$coordinate of the point to determine the equation of the parallel vertical line. The parallel line through (-1, 3) would be $x = -1$.

3. First, find the slope between (6, -1) and (1, 3).

$$m = \frac{-1-3}{6-1} = \frac{-4}{5} = -\frac{4}{5}$$

This will also be the slope of the parallel line. Use this slope with the given point, (-5, 2).

$$\begin{aligned} 2 &= -\frac{4}{5}(-5) + b \\ 2 &= 1 + b \\ 1 &= b \end{aligned}$$

The equation of the parallel line is $y = -\frac{4}{5}x + 1$.

Vocabulary

Parallel
: When two or more lines are in the same plane and never intersect. These lines will always have the same slope.

Problem Set

Find the equation of the line, given the following information. You may leave your answer in slope-intercept form.

1. Passes through (4, 7) and is parallel to $x-y=-5$.
2. Passes through (-6, -2) and is parallel to $y=4$.
3. Passes through (-3, 5) and is parallel to $y=-\frac{1}{3}x-1$.
4. Passes through (1, -9) and is parallel to $x=8$.
5. Passes through the $y-$intercept of $2x-3y=6$ and parallel to $x-4y=10$.
6. Passes through (-12, 4) and is parallel to $y=-3x+5$.
7. Passes through the $x-$intercept of $2x-3y=6$ and parallel to $x+4y=-3$.
8. Passes through (7, -8) and is parallel to $2x+5y=14$.
9. Passes through (1, 3) and is parallel to the line that passes through (-6, 2) and (-4, 6).
10. Passes through (-18, -10) and is parallel to the line that passes through (-2, 2) and (-8, 1).
11. Passes through (-4, -1) and is parallel to the line that passes through (15, 7) and (-1, -1).

Are the pairs of lines parallel? Briefly explain how you know.

12. $x-2y=4$ and $-5x+10y=16$
13. $3x+4y=-8$ and $6x+12y=-1$
14. $5x-5y=20$ and $x+y=7$
15. $8x-12y=36$ and $10x-15y=-15$

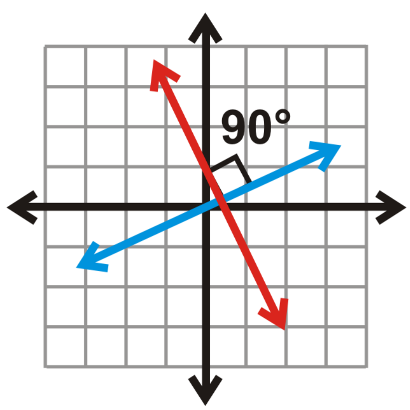
Finding the Equation of Perpendicular Lines

Objective

To find the equation of a line that is perpendicular to a given line and determine if pairs of lines are parallel, perpendicular, or neither.

Guidance

When two lines are perpendicular, they intersect at a 90°, or right, angle. The slopes of two perpendicular lines, are therefore, not the same. Let's investigate the relationship of perpendicular lines.



Investigation: Slopes of Perpendicular Lines

Tools Needed: Pencil, ruler, protractor, and graph paper

1. Draw an $x-y$ plane that goes from -5 to 5 in both the x and y directions.
2. Plot (0, 0) and (1, 3). Connect these to form a line.
3. Plot (0, 0) and (-3, 1). Connect these to form a second line.
4. Using a protractor, measure the angle formed by the two lines. What is it?

5. Use slope triangles to find the slope of both lines. What are they?
6. Multiply the slope of the first line times the slope of the second line. What do you get?

From this investigation, the lines from #2 and #3 are perpendicular because they form a 90° angle. The slopes are 3 and $-\frac{1}{3}$, respectively. When multiplied together, the product is -1. This is true of all perpendicular lines.

The product of the slopes of two perpendicular lines is -1.

If a line has a slope of m, *then the perpendicular slope is* $-\frac{1}{m}$.

Example A

Find the equation of the line that is perpendicular to $2x-3y=15$ and passes through (6, 5).

Solution: First, we need to change the line from standard to slope-intercept form.

$$\begin{aligned}2x-3y&=15\\-3y&=-2x+15\\y&=\frac{2}{3}x-5\end{aligned}$$

Now, let's find the perpendicular slope. From the investigation above, we know that the slopes must multiply together to equal -1.

$$\begin{aligned}\frac{2}{3}\cdot m&=-1\\\frac{\cancel{3}}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot m&=-1\cdot\frac{3}{2}\\m&=-\frac{3}{2}\end{aligned}$$

Notice that the perpendicular slope is the *opposite sign and reciprocals* with the original slope. Now, we need to use the given point to find the $y-$intercept.

$$\begin{aligned}5&=-\frac{3}{2}(6)+b\\5&=-9+b\\14&=b\end{aligned}$$

The equation of the line that is perpendicular to $y=\frac{2}{3}x-5$ is $y=-\frac{3}{2}x+14$.

If we write these lines in standard form, the equations would be $2x-3y=15$ and $3x+2y=28$, respectively.

Example B

Write the equation of the line that passes through (4, -7) and is perpendicular to $y=2$.

Solution: The line $y=2$ does not have an $x-$term, meaning it has no slope and a horizontal line. Therefore, to find the perpendicular line that passes through (4, -7), it would have to be a vertical line. Only need the $x-$coordinate. The perpendicular line would be $x=4$.

Example C

Write the equation of the line that passes through (6, -10) and is perpendicular to the line that passes through (4, -6) and (3, -4).

Solution: First, we need to find the slope of the line that our line will be perpendicular to. Use the points (4, -6) and (3, -4) to find the slope.

$$m = \frac{-4-(-6)}{3-4} = \frac{2}{-1} = -2$$

Therefore, the perpendicular slope is the opposite sign and the reciprocal of -2. That makes the slope $\frac{1}{2}$. Use the point (6, -10) to find the $y-$intercept.

$$\begin{aligned}-10 &= \frac{1}{2}(6)+b\\ -10 &= 3+b\\ -7 &= b\end{aligned}$$

The equation of the perpendicular line is $y = \frac{1}{2}x-7$.

Guided Practice

1. Find the equation of the line that is perpendicular to $x-2y=8$ and passes through (4, -3).

2. Find the equation of the line that passes through (-8, 7) and is perpendicular to the line that passes through (6, -1) and (1, 3).

3. Are $x-4y=8$ and $2x+8y=-32$ parallel, perpendicular or neither?

Answers

1. First, we need to change this line from standard form to slope-intercept form.

$$\begin{aligned}x-2y &= 8\\ -2y &= -x+8\\ y &= \frac{1}{2}x-4\end{aligned}$$

The perpendicular slope will be -2. Let's find the new $y-$intercept.

$$\begin{aligned}-3 &= -2(4)+b\\ -3 &= -8+b\\ 5 &= b\end{aligned}$$

The equation of the perpendicular line is $y=-2x+5$ or $2x+y=5$.

2. First, find the slope between (6, -1) and (1, 3).

$$m = \frac{-1-3}{6-1} = \frac{-4}{5} = -\frac{4}{5}$$

From this, the slope of the perpendicular line will be $\frac{5}{4}$. Now, use (-8, 7) to find the $y-$intercept.

$$\begin{aligned}7 &= \frac{5}{4}(-8)+b \\ 7 &= -10+b \\ 17 &= b\end{aligned}$$

The equation of the perpendicular line is $y = \frac{5}{4}x+17$.

3. To determine if the two lines are parallel or perpendicular, we need to change them both into slope-intercept form.

$$\begin{aligned}x-4y &= 8 & \qquad 2x+8y &= -32 \\ -4y &= -x+8 \quad \textit{and} & 8y &= -2x-32 \\ y &= \frac{1}{4}x-2 & y &= -\frac{1}{4}x-4\end{aligned}$$

Now, just look at the slopes. One is $\frac{1}{4}$ and the other is $-\frac{1}{4}$. They are not the same, so they are not parallel. To be perpendicular, the slopes need to be reciprocals, which they are not. Therefore, these two lines are not parallel or perpendicular.

Vocabulary

Perpendicular
: When two lines intersect to form a right, or 90°, angle. The product of the slopes of two perpendicular lines is -1.

Problem Set

Find the equation of the line, given the following information. You may leave your answer in slope-intercept form.

1. Passes through (4, 7) and is perpendicular to $x-y=-5$.
2. Passes through (-6, -2) and is perpendicular to $y=4$.
3. Passes through (4, 5) and is perpendicular to $y=-\frac{1}{3}x-1$.
4. Passes through (1, -9) and is perpendicular to $x=8$.
5. Passes through (0, 6) and perpendicular to $x-4y=10$.
6. Passes through (-12, 4) and is perpendicular to $y=-3x+5$.
7. Passes through the $x-$intercept of $2x-3y=6$ and perpendicular to $x+6y=-3$.
8. Passes through (7, -8) and is perpendicular to $2x+5y=14$.
9. Passes through (1, 3) and is perpendicular to the line that passes through (-6, 2) and (-4, 6).
10. Passes through (3, -10) and is perpendicular to the line that passes through (-2, 2) and (-8, 1).
11. Passes through (-4, -1) and is perpendicular to the line that passes through (-15, 7) and (-3, 3).

Are the pairs of lines parallel, perpendicular or neither?

12. $4x+2y=5$ and $5x-10y=-20$
13. $9x+12y=8$ and $6x+8y=-1$
14. $5x-5y=20$ and $x+y=7$
15. $8x-4y=12$ and $4x-y=-15$

3.8 Graphing Lines

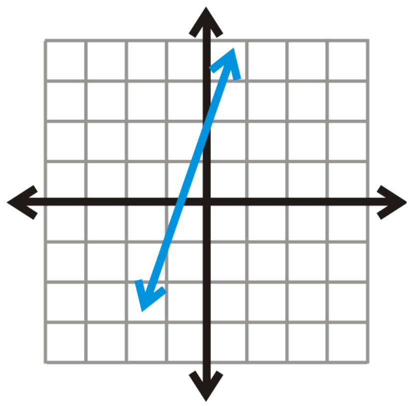
Objective

To be able to graph the equation of a line in slope-intercept or standard form.

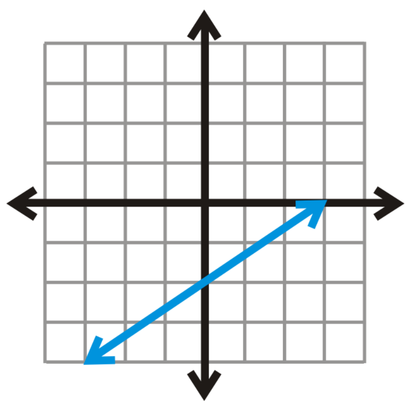
Review Queue

Find the equation of each line below. For the graphs, you may assume the $y-$intercepts are integers.

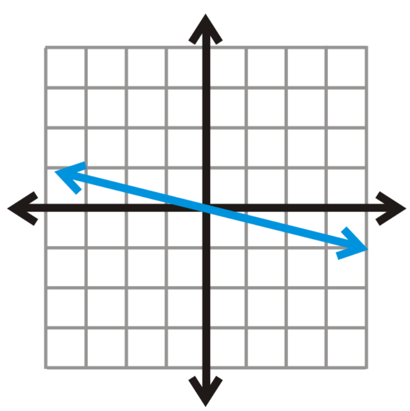
1.



2.



3.



4. What are the x and $y-$intercepts of:

a) $3x-5y=15$

b) $8x-5y=24$

Graph a Line in Slope-Intercept Form

Objective

To graph a line in slope-intercept form.

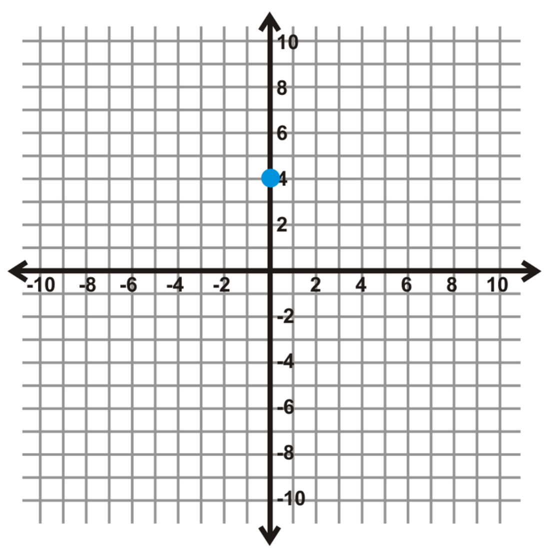
Guidance

From the previous lesson, we know that the equation of a line is $y=mx+b$, where m is the slope and b is the $y-$intercept. From these two pieces of information we can graph any line.

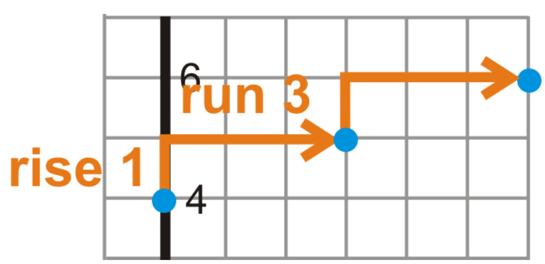
Example A

Graph $y=\frac{1}{3}x+4$ on the Cartesian plane.

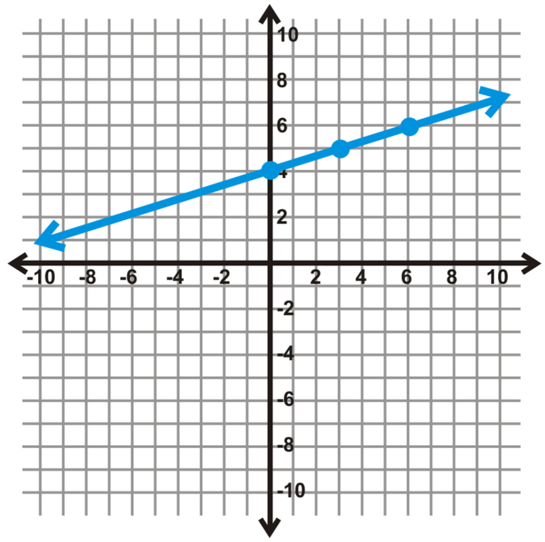
Solution: First, the Cartesian plane is the $x-y$ plane. Typically, when graphing lines, draw each axis from -10 to 10. To graph this line, you need to find the slope and $y-$intercept. By looking at the equation, $\frac{1}{3}$ is the slope and 4, or (0, 4), is the $y-$intercept. To start graphing this line, plot the $y-$intercept on the $y-$axis.



Now, we need to use the slope to find the next point on the line. Recall that the slope is also $\frac{rise}{run}$, so for $\frac{1}{3}$, we will rise 1 and run 3 from the $y-$intercept. Do this a couple of times to get at least three points.



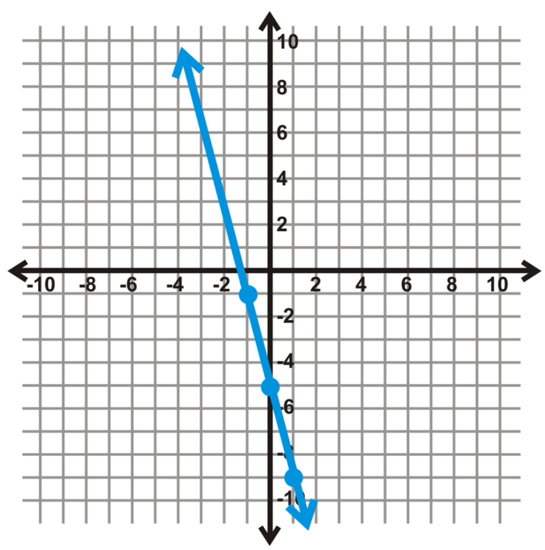
Now that we have three points, connect them to form the line $y=\frac{1}{3}x+4$.



Example B

Graph $y = -4x - 5$.

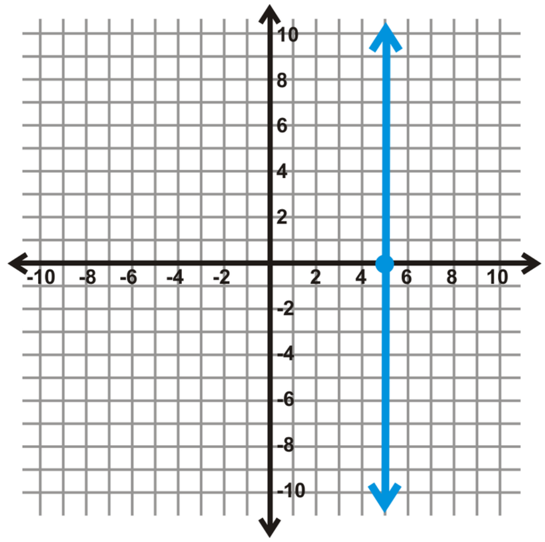
Solution: Now that the slope is negative, the vertical distance will "fall" instead of rise. Also, because the slope is a whole number, we need to put it over 1. Therefore, for a slope of -4, the line will fall 4 and run 1 OR rise 4 and run backward 1. Start at the $y-$intercept, and then use the slope to find a few more points.



Example C

Graph $x = 5$.

Solution: Any line in the form $x = a$ is a vertical line. To graph any vertical line, plot the value, in this case 5, on the $x-$axis. Then draw the vertical line.



To graph a horizontal line, $y = b$, it will be the same process, but plot the value given on the $y-$axis and draw a horizontal line.

Guided Practice

Graph the following lines.

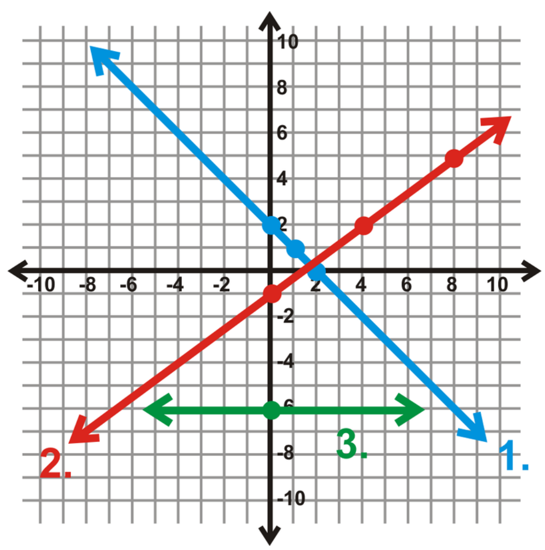
1. $y = -x + 2$

2. $y = \frac{3}{4}x - 1$

3. $y = -6$

Answers

All the answers are on the same grid below.



1. Plot (0, 2) and the slope is -1, which means you fall 1 and run 1.

2. Plot (0, -1) and then rise 3 and run 4 to the next point, (4, 2).

3. Plot -6 on the $y-$axis and draw a horizontal line.

Problem Set

Graph the following lines in the Cartesian plane.

1. $y = -2x - 3$
2. $y = x + 4$
3. $y = \frac{1}{3}x - 1$
4. $y = 9$
5. $y = -\frac{2}{5}x + 7$
6. $y = \frac{2}{4}x - 5$
7. $y = -5x - 2$
8. $y = -x$
9. $y = 4$
10. $x = -3$
11. $y = \frac{3}{2}x + 3$
12. $y = -\frac{1}{6}x - 8$
13. Graph $y = 4$ and $x = -6$ on the same set of axes. Where do they intersect?
14. If you were to make a general rule for the lines $y = b$ and $x = a$, where will they always intersect?
15. The cost per month, C (in dollars), of placing an ad on a website is $C = 0.25x + 50$, where x is the number of times someone clicks on your link. How much would it cost you if 500 people clicked on your link?

Graph a Line in Standard Form

Objective

To graph a line in standard form.

Guidance

When a line is in standard form, there are two different ways to graph it. The first is to change the equation to slope-intercept form and then graph as shown in the previous concept. The second is to use standard form to find the x and $y-$intercepts of the line and connect the two. Here are a few examples.

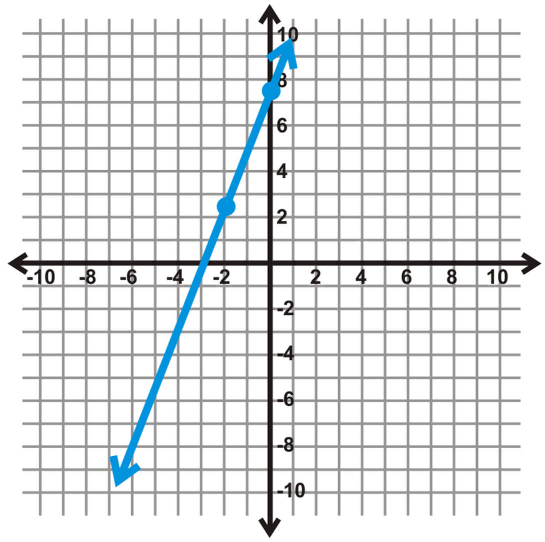
Example A

Graph $5x - 2y = -15$.

Solution: Let's use approach #1; change the equation to slope-intercept form.

$$\begin{aligned}
5x - 2y &= -15 \\
-2y &= -5x - 15 \\
y &= \frac{5}{2}x + \frac{15}{2}
\end{aligned}$$

The $y-$intercept is $\left(0, \frac{15}{2}\right)$. Change the improper fraction to a decimal and approximate it on the graph, (0, 7.5). Then use slope triangles. If you find yourself running out of room "rising 5" and "running 2," you could also "fall 5" and "run backwards 2" to find a point on the other side of the $y-$intercept.



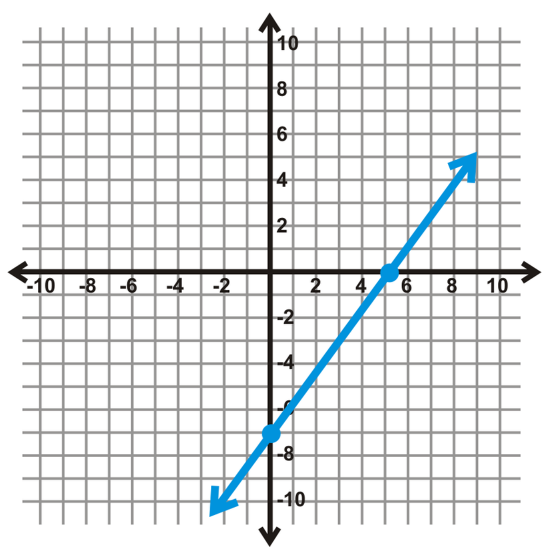
Example B

Graph $4x - 3x = 21$.

Solution: Let's use approach #2; find the x and $y-$intercepts (from *Standard Form of a Line* concept). Recall that the other coordinate will be zero at these points. Therefore, for the $x-$intercept, plug in zero for y and for the $y-$intercept, plug in zero for x.

$$\begin{aligned} 4x - 3(0) &= 21 \\ 4x &= 21 \\ x &= \frac{21}{4} \ or\ 5.25 \end{aligned} \qquad\qquad \begin{aligned} 4(0) - 3y &= 21 \\ -3y &= 21 \\ y &= -7 \end{aligned}$$

Now, plot each on their respective axes and draw a line.



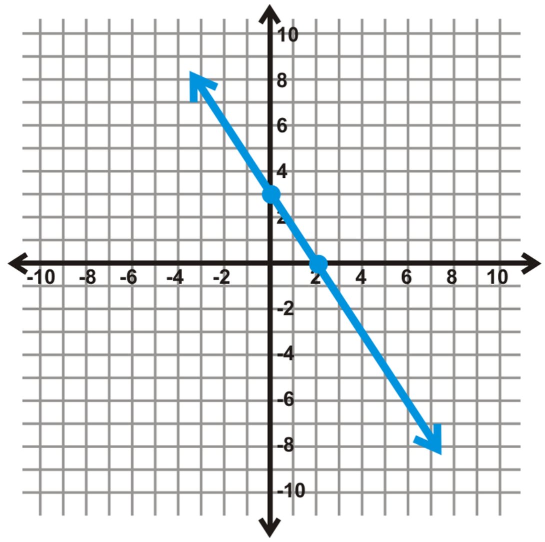
Guided Practice

1. Graph $4x + 6y = 18$ by changing it into slope-intercept form.

2. Graph $5x - 3y = 30$ by plotting the intercepts.

Answers

1. Change $4x+6y=18$ into slope-intercept form by solving for y, then graph.

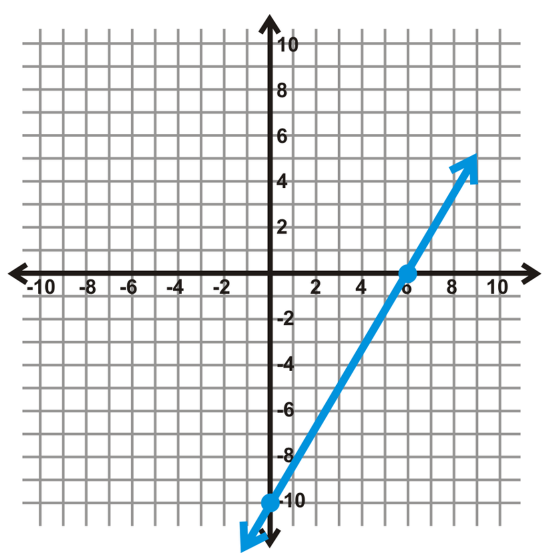


$$
\begin{aligned}
4x+6y &= 18 \\
6y &= -4x+18 \\
y &= -\frac{2}{3}x+3
\end{aligned}
$$

2. Substitute in zero for x, followed by y and solve each equation.

$$
\begin{aligned}
5(0)-3y &= 30 \\
-3y &= 30 \\
y &= -10
\end{aligned}
\qquad\qquad
\begin{aligned}
5x-3(0) &= 30 \\
5x &= 30 \\
x &= 6
\end{aligned}
$$

Now, plot each on their respective axes and draw a line.



Problem Set

Graph the following lines by changing the equation to slope-intercept form.

1. $-2x+y=5$
2. $3x+8y=16$
3. $4x-2y=10$
4. $6x+5y=-20$
5. $9x-6y=24$
6. $x+4y=-12$

Graph the following lines by finding the intercepts.

7. $2x+3y=12$
8. $-4x+5y=30$
9. $x-2y=8$
10. $7x+y=-7$
11. $6x+10y=15$
12. $4x-8y=-28$
13. **Writing** Which method do you think is easier? Why?
14. **Writing** Which method would you use to graph $x=-5$? Why?

3.9 Parallel and Perpendicular Lines in the Coordinate Plane

Learning Objectives

- Compute slope.
- Determine the equation of parallel and perpendicular lines to a given line.
- Graph parallel and perpendicular lines in slope-intercept and standard form.

Review Queue

Find the slope between the following points.

1. (-3, 5) and (2, -5)

2. (7, -1) and (-2, 2)

3. Is $x = 3$ horizontal or vertical? How do you know?

Graph the following lines on an $x-y$ plane.

4. $y = -2x+3$

5. $y = \frac{1}{4}x-2$

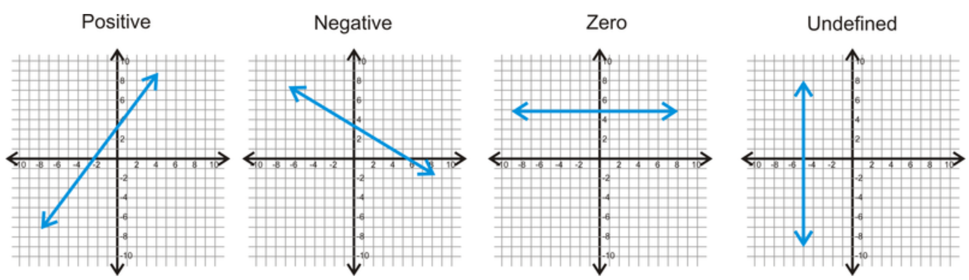
Know What? The picture to the right is the California Incline, a short piece of road that connects Highway 1 with the city of Santa Monica. The length of the road is 1532 feet and has an elevation of 177 feet. *You may assume that the base of this incline is sea level, or zero feet.* **Can you find the slope of the California Incline?**

HINT: You will need to use the Pythagorean Theorem, which has not been introduced in this class, but you may have seen it in a previous math class.

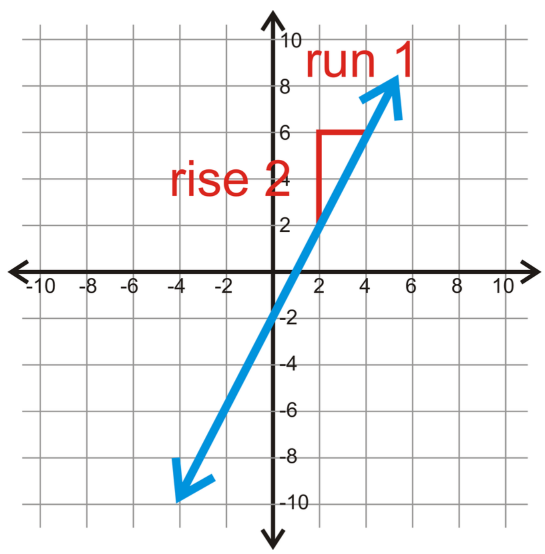
Slope in the Coordinate Plane

Recall from Algebra I, The slope of the line between two points (x_1, y_1) and (x_2, y_2) is $m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$.

Different Types of Slope:



Example 1: What is the slope of the line through (2, 2) and (4, 6)?



Solution: Use the slope formula to determine the slope. Use (2, 2) as (x_1, y_1) and (4, 6) as (x_2, y_2).

$$m = \frac{6-2}{4-2} = \frac{4}{2} = 2$$

Therefore, the slope of this line is 2.

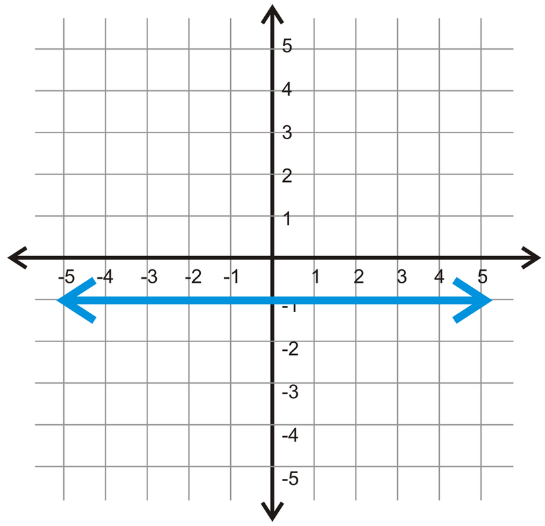
This slope is positive. Recall that slope can also be the "rise over run." In this case we "rise", or go up 2, and "run" in the positive direction 1.

Example 2: Find the slope between (-8, 3) and (2, -2).

Solution: $m = \frac{-2-3}{2-(-8)} = \frac{-5}{10} = -\frac{1}{2}$

This is a negative slope. Instead of "rising," the negative slope means that you would "fall," when finding points on the line.

Example 3: Find the slope between (-5, -1) and (3, -1).

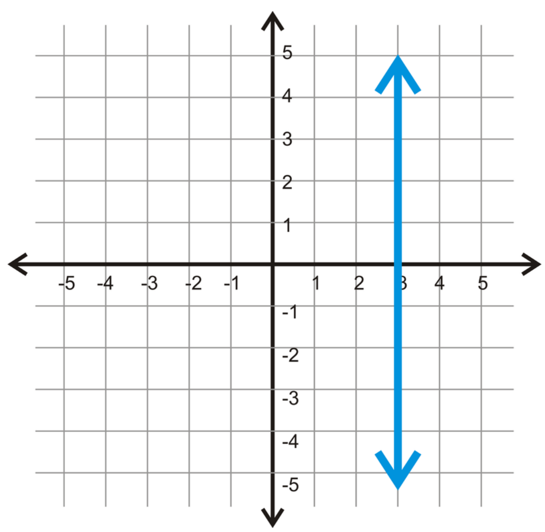


Solution:

$$m = \frac{-1-(-1)}{3-(-5)} = \frac{0}{8} = 0$$

Therefore, the slope of this line is 0, which means that it is a horizontal line. Horizontallines always pass through the $y-$axis. Notice that the $y-$coordinate for both points is -1. In fact, the $y-$coordinate for *any* point on this line is -1. This means that the horizontal line must cross $y = -1$.

Example 4: What is the slope of the line through (3, 2) and (3, 6)?



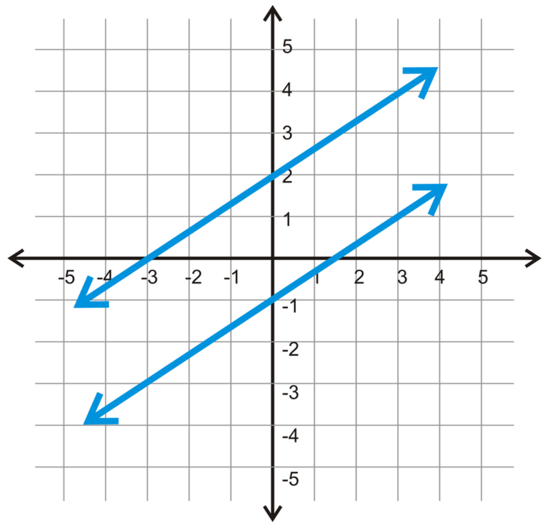
Solution:

$$m = \frac{6-2}{3-3} = \frac{4}{0} = undefined$$

Therefore, the slope of this line is undefined, which means that it is a *vertical* line. Verticallines always pass through the $x-$axis. Notice that the $x-$coordinate for both points is 3. In fact, the $x-$coordinate for *any* point on this line is 3. This means that the vertical line must cross $x = 3$.

Slopes of Parallel Lines

Recall from earlier in the chapter that the definition of parallel is two lines that never intersect. In the coordinate plane, that would look like this:



If we take a closer look at these two lines, we see that the slopes of both are $\frac{2}{3}$.

This can be generalized to any pair of parallel lines in the coordinate plane.

Parallel lines have the same slope.

Example 5: Find the equation of the line that is parallel to $y = -\frac{1}{3}x + 4$ and passes through (9, -5).

Recall that the equation of a line in this form is called the slope-intercept form and is written as $y = mx + b$ where m is the slope and b is the $y-$intercept. Here, x and y represent any coordinate pair, (x, y) on the line.

Solution: We know that parallel lines have the same slope, so the line we are trying to find also has $m = -\frac{1}{3}$. Now, we need to find the $y-$intercept. 4 is the $y-$intercept of the given line, *not our new line*. We need to plug in 9 for x and -5 for y (this is our given coordinate pair that needs to be on the line) to solve for the *new* $y-$intercept (b).

$$-5 = -\frac{1}{3}(9) + b$$

$$-5 = -3 + b \qquad \text{Therefore, the equation of line is } y = -\frac{1}{3}x - 2.$$

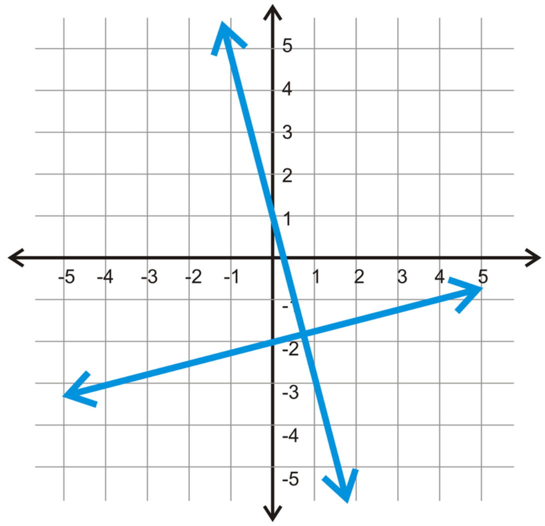
$$-2 = b$$

Reminder: the final equation contains the variables x and y to indicate that the line contains and infinite number of points or coordinate pairs that satisfy the equation.

Parallel lines always have the <u>same slope</u> and <u>different $y-$intercepts</u>.

Slopes of Perpendicular Lines

Recall from Chapter 1 that the definition of perpendicular is two lines that intersect at a 90°, or right, angle. In the coordinate plane, that would look like this:



If we take a closer look at these two lines, we see that the slope of one is -4 and the other is $\frac{1}{4}$.

This can be generalized to any pair of perpendicular lines in the coordinate plane.

The slopes of perpendicular lines are opposite signs and reciprocals of each other.

Example 6: Find the slope of the perpendicular lines to the lines below.

a) $y = 2x+3$

b) $y = -\frac{2}{3}x-5$

c) $y = x+2$

Solution: We are only concerned with the slope for each of these.

a) $m = 2$, so $m_\perp$ is the reciprocal and negative, $m_\perp = -\frac{1}{2}$.

b) $m = -\frac{2}{3}$, take the reciprocal and make the slope positive, $m_\perp = \frac{3}{2}$.

c) Because there is no number in front of x, the slope is 1. The reciprocal of 1 is 1, so the only thing to do is make it negative, $m_\perp = -1$.

Example 7: Find the equation of the line that is perpendicular to $y = -\frac{1}{3}x+4$ and passes through (9, -5).

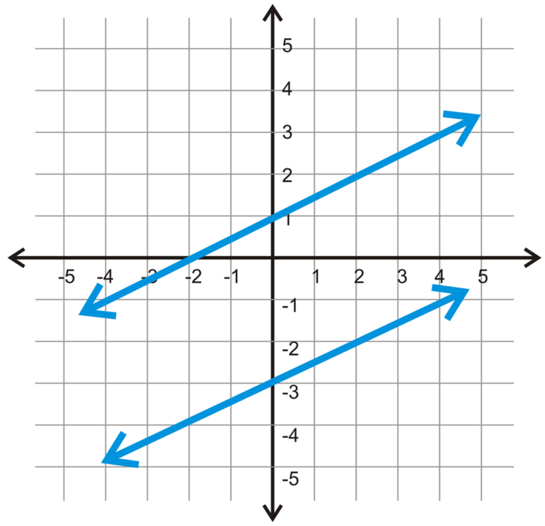
Solution: First, the slope is the reciprocal and opposite sign of $-\frac{1}{3}$. So, $m = 3$. Now, we need to find the $y-$intercept. 4 is the $y-$intercept of the given line, *not our new line*. We need to plug in 9 for x and -5 for y to solve for the *new* $y-$intercept (b).

$$\begin{aligned} -5 &= 3(9)+b \\ -5 &= 27+b \\ -32 &= b \end{aligned}$$

Therefore, the equation of line is $y = 3x-32$.

Graphing Parallel and Perpendicular Lines

Example 8: Find the equations of the lines below and determine if they are parallel, perpendicular or neither.



Solution: To find the equation of each line, start with the $y-$intercept. The top line has a $y-$intercept of 1. From there, determine the slope triangle, or the "rise over run." From the $y-$intercept, if you go up 1 and over 2, you hit the line again. Therefore, the slope of this line is $\frac{1}{2}$. The equation is $y = \frac{1}{2}x + 1$. For the second line, the $y-$intercept is -3. Again, start here to determine the slope and if you "rise" 1 and "run" 2, you run into the line again, making the slope $\frac{1}{2}$. The equation of this line is $y = \frac{1}{2}x - 3$. The lines are parallel because they have the same slope.

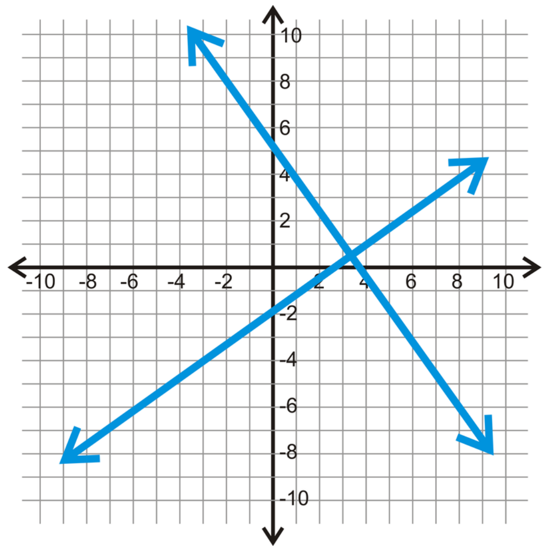
Example 9: Graph $3x - 4y = 8$ and $4x + 3y = 15$. Determine if they are parallel, perpendicular, or neither.

Solution: First, we have to change each equation into slope-intercept form. In other words, we need to solve each equation for y.

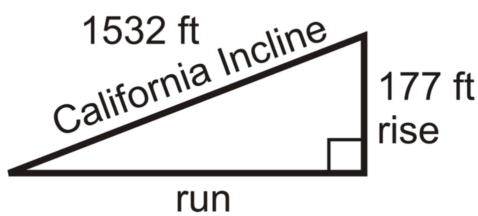
$$\begin{aligned} 3x - 4y &= 8 \\ -4y &= -3x + 8 \\ y &= \frac{3}{4}x - 2 \end{aligned} \qquad\qquad \begin{aligned} 4x + 3y &= 15 \\ 3y &= -4x + 15 \\ y &= -\frac{4}{3}x + 5 \end{aligned}$$

Now that the lines are in slope-intercept form (also called $y-$intercept form), we can tell they are perpendicular because the slopes are opposites signs and reciprocals.

To graph the two lines, plot the $y-$intercept on the $y-$axis. From there, use the slope to rise and then run. For the first line, you would plot -2 and then rise 3 and run 4, making the next point on the line (1, 4). For the second line, plot 5 and then fall (because the slop is negative) 4 and run 3, making the next point on the line (1, 3).



Know What? Revisited In order to find the slope, we need to first find the horizontal distance in the triangle to the right. This triangle represents the incline and the elevation. To find the horizontal distance, or the run, we need to use the Pythagorean Theorem, $a^2 + b^2 = c^2$, where c is the hypotenuse.



$$\begin{aligned} 177^2 + run^2 &= 1532^2 \\ 31,329 + run^2 &= 2,347,024 \\ run^2 &= 2,315,695 \\ run &\approx 1521.75 \end{aligned}$$

The slope is then $\frac{177}{1521.75}$, which is roughly $\frac{3}{25}$.

Review Questions

Find the slope between the two given points.

1. (4, -1) and (-2, -3)
2. (-9, 5) and (-6, 2)
3. (7, 2) and (-7, -2)
4. (-6, 0) and (-1, -10)
5. (1, -2) and (3, 6)
6. (-4, 5) and (-4, -3)

Determine if each pair of lines are parallel, perpendicular, or neither. Then, graph each pair on the same set of axes.

7. $y = -2x + 3$ and $y = \frac{1}{2}x + 3$
8. $y = 4x - 2$ and $y = 4x + 5$
9. $y = -x + 5$ and $y = x + 1$
10. $y = -3x + 1$ and $y = 3x - 1$
11. $2x - 3y = 6$ and $3x + 2y = 6$
12. $5x + 2y = -4$ and $5x + 2y = 8$
13. $x - 3y = -3$ and $x + 3y = 9$
14. $x + y = 6$ and $4x + 4y = -16$

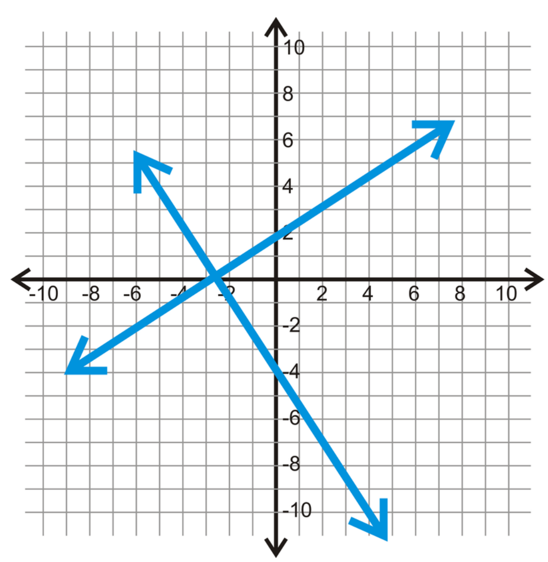
Determine the equation of the line that is ***parallel*** to the given line, through the given point.

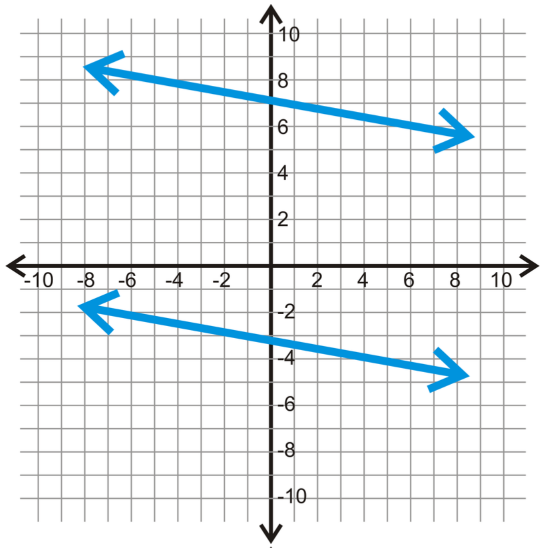
15. $y = -5x + 1;\ (-2,\ 3)$
16. $y = \frac{2}{3}x - 2;\ (9, 1)$
17. $x - 4y = 12;\ (-16,\ -2)$
18. $3x + 2y = 10;\ (8,\ -11)$
19. $2x - y = 15;\ (3,\ 7)$
20. $y = x - 5;\ (9,\ -1)$

Determine the equation of the line that is ***perpendicular*** to the given line, through the given point.

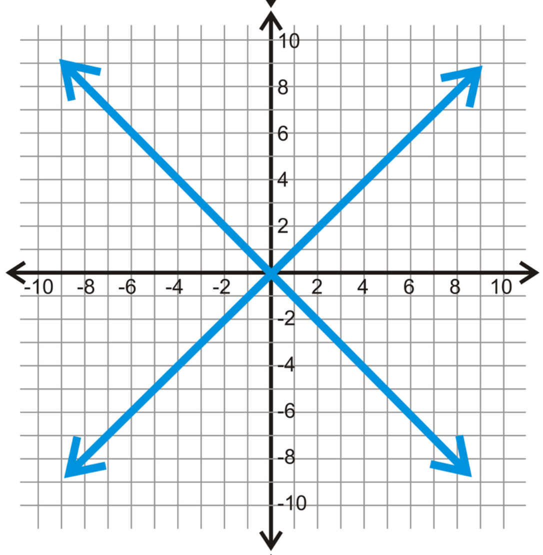
21. $y = x - 1;\ (-6,\ 2)$
22. $y = 3x + 4;\ (9,\ -7)$
23. $5x - 2y = 6;\ (5,\ 5)$
24. $y = 4;\ (-1,\ 3)$
25. $x = -3;\ (1,\ 8)$
26. $x - 3y = 11;\ (0,\ 13)$

Find the equation of the two lines in each graph below. Then, determine if the two lines are parallel, perpendicular or neither.

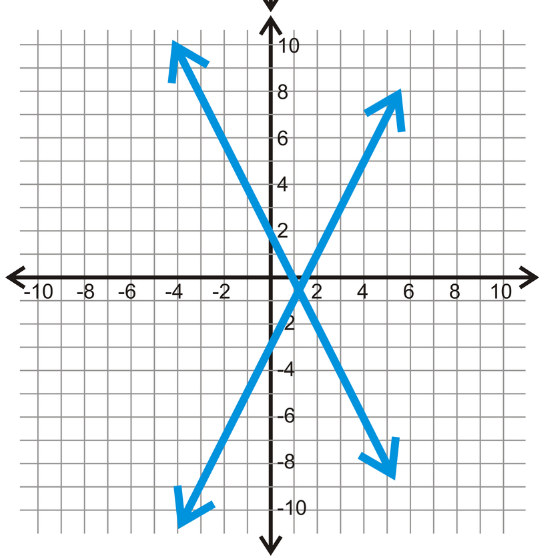
27. 



28.



29.

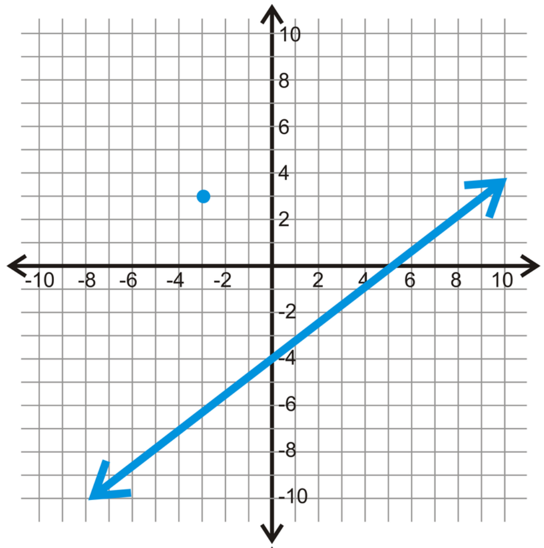


30.

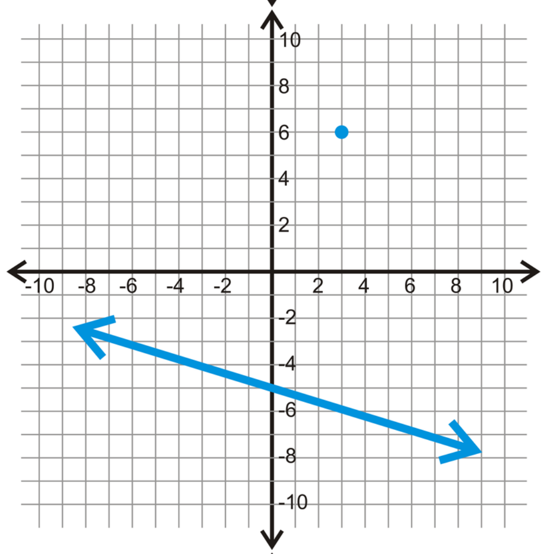
For the line and point below, find:

a) A parallel line, through the given point.

b) A perpendicular line, through the given point.


10
8
6
4
2
-10 -8 -6 -4 -2 2 4 6 8 10
-2
-4
-6
-8
-10

31.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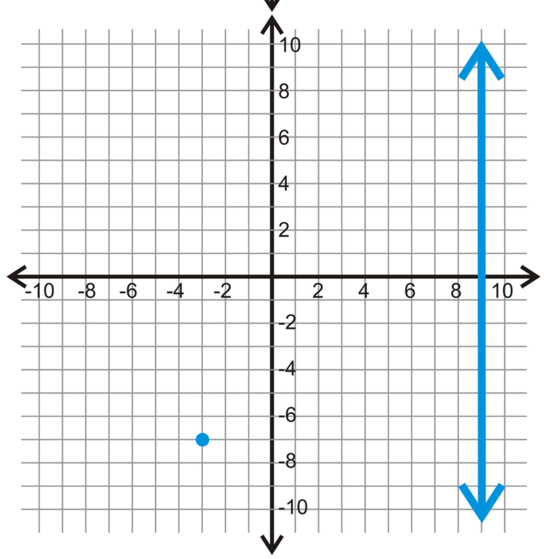
-8
-10

32.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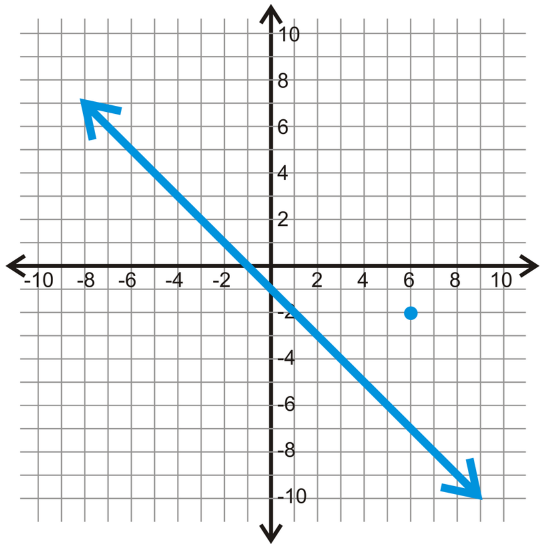
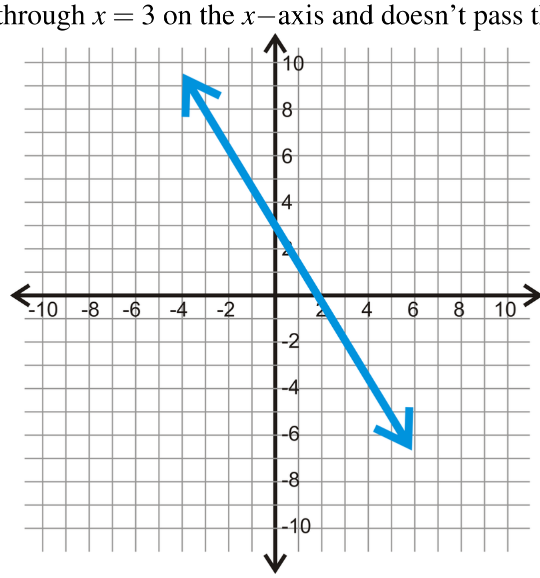
-4
-6
-8
-10

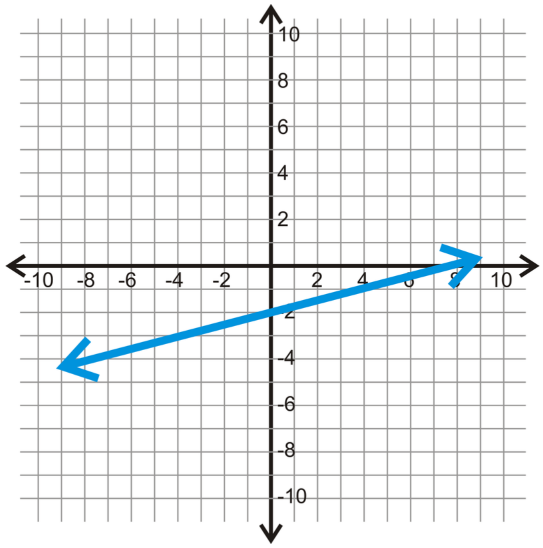
33.

34. 

Review Queue Answers

a. $m = \frac{-5-5}{2+3} = \frac{-10}{2} = -5$
b. $m = \frac{2+1}{-2-7} = \frac{3}{-9} = -\frac{1}{3}$
c. Vertical because it has to pass through $x = 3$ on the $x-$axis and doesn't pass through y at all.

d. 

e. 

3.10 Chapter 3 Review

Keywords and Theorems

Parallel
When two or more lines lie in the same plane and never intersect.

Skew Lines
Lines that are in different planes and never intersect.

Parallel Postulate
For a line and a point not on the line, there is exactly one line parallel to this line through the point.

Perpendicular Line Postulate
For a line and a point not on the line, there is exactly one line parallel to this line through the point.

Transversal
A line that intersects two distinct lines. These two lines may or may not be parallel.

Corresponding Angles
Two angles that are in the "same place" with respect to the transversal, but on different lines.

Alternate Interior Angles
Two angles that are on the interior of l and m, but on opposite sides of the transversal.

Alternate Exterior Angles
Two angles that are on the exterior of l and m, but on opposite sides of the transversal.

Same Side Interior Angles
Two angles that are on the same side of the transversal and on the interior of the two lines.

Corresponding Angles Postulate
If two parallel lines are cut by a transversal, then the corresponding angles are congruent.

Alternate Interior Angles Theorem
If two parallel lines are cut by a transversal, then the alternate interior angles are congruent.

Alternate Exterior Angles Theorem
If two parallel lines are cut by a transversal, then the alternate exterior angles are congruent.

Same Side Interior Angles Theorem
If two parallel lines are cut by a transversal, then the same side interior angles are supplementary.

Converse of Corresponding Angles Postulate
If corresponding angles are congruent when two lines are cut by a transversal, then the lines are parallel.

Converse of Alternate Interior Angles Theorem
If two lines are cut by a transversal and alternate interior angles are congruent, then the lines are parallel.

Converse of the Alternate Exterior Angles Theorem
If two lines are cut by a transversal and the alternate exterior angles are congruent, then the lines are parallel.

Converse of the Same Side Interior Angles Theorem
If two lines are cut by a transversal and the consecutive interior angles are supplementary, then the lines are parallel.

Parallel Lines Property
The Parallel Lines Property is a transitive property that can be applied to parallel lines.

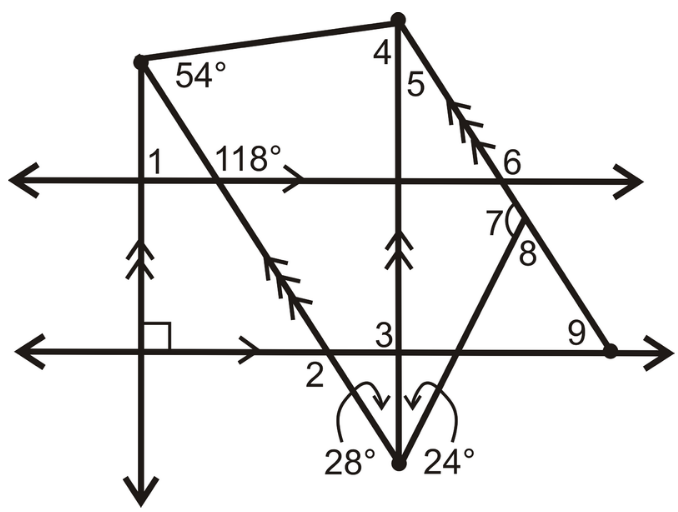
Theorem 3-1
If two lines are parallel and a third line is perpendicular to one of the parallel lines, it is also perpendicular to the other parallel line.

Theorem 3-2
If two lines are parallel and a third line is perpendicular to one of the parallel lines, it is also perpendicular to the other parallel line.

Distance Formula
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Review

Find the value of each of the numbered angles below.



Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9688 .